Polyspectral Factorization

Tucker McElroy[1] Dhrubajyoti Ghosh[2] Soumendra Lahiri[3]

[1]US Census Bureau

[2]Washington University

[3]Washington University

Abstract

This presentation contributes to the theoretical background for a new quadratic prediction method for time series. We develop a theory of polyspectral factorization, providing new mathematical results for polyspectral densities. New bijections between a restricted space of higher-dimensional cepstral coefficients (where the restrictions are induced by the symmetries of the polyspectra) and the auto-cumulants are derived. Applications to modeling are developed; in particular, it is shown that semi-parametric nonlinear time series modeling can be accomplished by approximation of the cepstral representation of polyspectra.

Disclaimer

This presentation is released to inform interested parties of research and to encourage discussion. The views expressed on statistical issues are those of the author and not those of the U.S. Census Bureau. All time series analyzed in this presentation are from public or external data sources.

Funding

This work partially supported by NSF research grant 1811998, 2019-2021 (DMS Co-PI). *Development of a general framework for nonlinear prediction using auto- cumulants: Theory, methodology and computation*.

Outline

- Introduction
- Definitions and Motivation
- Case of a Single Variable
- Cepstral Approximation
- Bispectral Factorization
- Trispectral Factorization
- Polyspectral Factorization
- Polyspectral Modeling

Introduction

Background

- For applications in forecasting, modeling, and signal extraction of stationary time series, it is important to be able to factorize *polyspectra* of order $k+1$.
- Basic case: $k = 1$ is the *spectral density*, and its so-called *spectral factorization* provides a *causal* representation of the time series, by which we can compute forecast filters and time series residuals.
- Spectral factorization factors a Laurent series into the product of two power series, in z and z^{-1}.
- Case of $k = 2$: the *bi-spectral density*, used to assess skewness in a process.
- Case of $k = 3$: the *tri-spectral density*, used to assess kurtosis in a process.

Introduction

Questions of Interest

1. Given the $k+1$th order polyspectra of a stationary time series, how do we determine its factorization in terms of power series?
2. Are there natural models for polyspectra that facilitate simulation, model-fitting, and residual analysis?

Introduction

Overview of Results

1. Review of $k = 1$ case.
2. Explicit polyspectral factorization for $k = 2, 3$.
3. Algorithmic approach for $k > 3$.
4. Discussion of modeling nonlinear processes.

Definitions and Motivation

Autocumulants Definition

- Let $\{X_t\}$ be a $k+1$th order stationary time series with $k+1$ moments for given $k \geq 1$.
- So the $k+1$th order moments are finite and the $k+1$th order autocumulant function is defined by

$$\gamma_{h_1,\ldots,h_k} = \text{cum}\left(X_{t+h_1}, \ldots, X_{t+h_k}, X_t\right).$$

- Further, assume that the autocumulant function is absolutely summable for $\underline{h} = [h_1, \ldots, h_k]' \in \mathbb{Z}^k$. These autocumulant functions are subject to various symmetries acting on the lattice $\mathbb{Z}^k$, as detailed in Berg (2008).

Definitions and Motivation

Polyspectrum Definition

- The absolute summability condition suffices to define the $k+1$th order polyspectral density

$$f(\lambda_1, \ldots, \lambda_k) = \sum_{\underline{h} \in \mathbb{Z}^k} \gamma_{\underline{h}} \exp\{-i \underline{h}' \underline{\lambda}\},$$

where we set $\underline{\lambda} = [\lambda_1, \ldots, \lambda_k]'$, and each of these are frequencies in $[-\pi, \pi]$.

Definitions and Motivation

Extension to $\mathbb{C}^k$

- Set $U = \{z \in \mathbb{C} : |z| < 1\}$, so that its boundary is the unit circle $\partial U = \{z \in \mathbb{C} : |z| = 1\}$.
- We view the polyspectral density as the restriction of a function defined on $\mathbb{C}^k$ to the k-torus ∂U^k.
- We define the extension to $\mathbb{C}^k$:

$$f(z_1, \ldots, z_k) = \sum_{\underline{h} \in \mathbb{Z}^k} \gamma_{\underline{h}} z_1^{h_1} \cdots z_k^{h_k}.$$

- This is called the $k+1$th order autocumulant generating function (acgf).
- Evaluating at $z_j = e^{-i\lambda_j}$ for $1 \leq j \leq k$ clearly yields its restriction to the polyspectral density.

Definitions and Motivation

Application of Linear Filter

- One can apply a linear filter $\Psi(B) = \sum_{j \in \mathbb{Z}} \psi_j B^j$ to the time series $\{X_t\}$, yielding a new $\{Y_t\}$ defined by

$$Y_t = \Psi(B) X_t.$$

- Let f_y and f_x denote acgf of order $k+1$ for the $\{Y_t\}$ and $\{X_t\}$ processes; then (see Theorem 2.8.1 of Brillinger (1981))

$$f_y(z_1, \ldots, z_k) = \prod_{j=1}^{k} \Psi(z_j)\, \Psi(z_1^{-1} \cdots z_k^{-1})\, f_x(z_1, \ldots, z_k). \quad (1)$$

Definitions and Motivation

The Case of i.i.d. Input

- If $\{X_t\}$ is i.i.d., then its autocumulant functions are zero except at the origin of the lattice, which means the acgf is constant for all k.
- More generally, a kth order *white noise* is a process with constant acgf of all orders $\leq k+1$; see Tekalp and Erdem (1989).
- Denoting the kth cumulant of X_t by μ_k,

$$f_y(z_1, \ldots, z_k) = \mu_{k+1} \prod_{j=1}^{k} \Psi(z_j)\, \Psi(z_1^{-1} \cdots z_k^{-1}). \qquad (2)$$

Definitions and Motivation

When Can We Factor the Polyspectrum?

- It follows from the Wold decomposition (McElroy and Politis 2020) that any non-deterministic stationary time series $\{Y_t\}$ can be expressed as $Y_t = \Psi(B)\, X_t$, where $\Psi(z)$ is a power series such that $\Psi(0) = 1$, and $\{X_t\}$ is a first-order white noise process.
- This means that when $k = 1$, any acgf can be written in the form (2).
- This is a restatement of the spectral factorization theorem.
- What about when $k > 1$? Tekalp and Erdem (1989) provides necessary and sufficient conditions for such a factorization. (Although $\Psi(z)$ will be Laurent, not a power series.) Many processes of interest don't satisfy these conditions.

Definitions and Motivation

Why Do We Want a Factorization?

- Polyspectra describe a stationary nonlinear process through the autocumulant structure.
- These polyspectral functions have constraints are their form. E.g., for $k = 1$ they are real-valued on ∂U and non-negative.
- If we can factor the acgf in terms of power series, each of whose coefficients are any real number, then we can parametrically describe nonlinear processes. This gives a semi-parametric description of nonlinear time series.
- Nonlinear forecasting filters require polyspectral factorization.
- We obtain a general factorization involving causal functions (power series in several variables).

The Case of a Single Variable

Background Results

- Here we present some foundational results for the case $k = 1$ of the acgf; much of this material is not novel (we draw heavily from Rudin (1987) and Ahlfors (1979)). But it helps to set up the framework.
- We want to describe the causal and anti-causal portions of a Laurent series.
- How is the zero-pole structure of a function related to its Laurent series? How is this structure related to the coefficients of the Laurent series?

The Case of a Single Variable

From Circle to Annulus

- Case $k = 1$: if autocovariances are absolutely summable, then spectral density $f(z)$ is well-defined on unit circle ($z \in \partial U$).
- How do we extend the acgf from ∂U to an annulus?
- For any region (an open, simply connected set) $\Omega \in \mathbb{C}$, let $H(\Omega)$ denote the set of functions that are analytic (holomorphic) on Ω.
- Let an open disk of radius r centered at $a \in \mathbb{C}$ be denoted $D_r(a)$ (so $U = D_1(0)$).
- Because $f(z)$ is an acgf, it can be written as

$$f(z) = \sum_{h \geq 0} \gamma_h z^h + \sum_{h < 0} \gamma_h z^h = [f]_+(z) + [f]_-(z).$$

The Case of a Single Variable

From Circle to Annulus

- Since $[f]_+(z)$ is a power series, there is some radius of convergence R_+ such that $[f]_+ \in H(D_{R_+}(0))$.
- If the autocumulants (which are autocovariances in this case) have geometric decay, $R_+ > 1$.
- Similarly, $[f]_-(z^{-1})$ is also a power series, and hence has a radius of convergence $R_- > 1$.
- By flipping z, we see that $[f]_-(z)$ is analytic in $\mathbb{C} \setminus \overline{D_{1/R_-}(0)}$.
- Therefore, on the intersection of these two domains, namely $A = \{z \in \mathbb{C} : 1/R_- < |z| < R_+\}$, the function $f(z)$ is analytic.

The Case of a Single Variable

From Meromorphic to Laurent

- There is a converse to this result: suppose that f is analytic on an annulus A that contains the unit circle. Then f can be expressed as a Laurent series with well-defined coefficients γ_h, which have geometric decay.
- We show this by summarizing the constructive argument in Ahlfors (1979) (pp. 184-186).
- Define a curve Γ_+ via the path re^{it} for some $r < R_+$ (which is the outer radius of the annulus A) and $t \in [0, 2\pi]$. This is called a *positively oriented circle*, because as t increases we move counter-clockwise.
- Also let Γ_- be a *negatively oriented circle* defined via the path re^{-it} (moving clockwise) with $r > 1/R_-$ (the inner radius of the annulus).

The Case of a Single Variable

From Meromorphic to Laurent

$$\frac{1}{2\pi i} \oint_{\Gamma_+} f(y)(y-z)^{-1}\, dy \tag{3}$$

for $z \in D_{R_+}(0)$ is convergent and analytic.
Call this function $[f]_+(z)$ by definition.

$$\frac{1}{2\pi i} \oint_{\Gamma_-} f(y)(y-z)^{-1}\, dy \tag{4}$$

for $z \in \mathbb{C} \setminus \overline{D_{1/R_-}(0)}$ is analytic. Call this function $[f]_-(z)$ by definition.

The Case of a Single Variable

From Meromorphic to Laurent

- So $[f]_-(z^{-1})$ is analytic in $D_{R_-}(0)$, and hence has a power series expansion.
- The sum of $[f]_+(z)$ and $[f]_-(z)$ is $f(z)$ on the annulus, which follows from the Cauchy integral formula.
- The coefficients of the two power series expansions can then be used to compute the γ_h coefficients, and Hadamard's formula shows they must have geometric decay.
- **Summary**: every Laurent series with geometrically decaying coefficients corresponds to a function that is analytic on an annulus containing the unit circle, and conversely.

The Case of a Single Variable

Zeroes and Poles

- By 10.18 in Rudin (1987), such functions have a discrete zero set (i.e., the set $Z(f) = \{z \in \mathbb{C} : f(z) = 0\}$ is finite or countable), and are meromorphic on $\mathbb{C} \setminus \{0\}$.
- There can be an essential singularity at $z = 0$, but otherwise the set of poles $P(f) = \{z \in \mathbb{C} : 1/f(z) = 0\}$ is discrete.
- Excluding functions with an essential singularity at $z = 0$, we obtain a class $\mathcal{L}$ of functions denoted as *unit circle Laurent series*.

The Case of a Single Variable

An Example

- Consider $f(z) = (7 - 4z)/(3 - 7z + 2z^2)$.
- This can be decomposed into $f(z) = (3 - z)^{-1} + (1/2 - z)^{-1}$, so that there are poles at $z = 1/2, 3$.
- The annulus of convergence has outer radius $R_+ = 3$ and inner radius $1/R_- = 1/2$. By computing line integrals, we see that

$$[f]_+(z) = \frac{1}{3 - z} \qquad [f]_-(z) = \frac{1}{1/2 - z}.$$

- Expanding the two power series, we obtain $\gamma_h = (1/3)^{h+1}$ for $h \geq 0$ and $\gamma_h = -2^{h+1}$ for $h \leq -1$.
  In this case, there is no essential singularity at $z = 0$.

The Case of a Single Variable

The Laurent Series Coefficients

- The line integral over the unit circle is well-defined for functions in $\mathcal{L}$, and for any $h \in \mathbb{Z}$ we can define

$$\langle z^{-h} f(z) \rangle_z = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{ih\lambda} f(e^{-i\lambda}) \, d\lambda, \tag{5}$$

  which we claim is the hth Laurent coefficient.
- It can be calculated by a line integral over a path on ∂U that is either positively oriented (denoted ∂U_+) or negatively oriented (denoted ∂U_-):

$$\langle z^{-h} f(z) \rangle_z = \begin{cases} \frac{1}{2\pi i} \oint_{\partial U_+} z^{-h-1} f(z) \, dz \\ \frac{1}{2\pi i} \oint_{\partial U_-} z^{h-1} f(z^{-1}) \, dz \end{cases} \tag{6}$$

The Case of a Single Variable

The Laurent Series Coefficients

- E.g., if $h \leq -1$ use the first formula; by the Cauchy Residue Theorem (see 10.42 of Rudin (1987)) we compute the sum of the residues of the function $z^{-h-1} f(z)$ at the poles of f lying within U.
- For any $-\infty \leq r \leq s \leq \infty$, let $[f(z)]_r^s = \sum_{h=r}^s \gamma_h z^j$, where $\gamma_h = \langle z^{-h} f(z) \rangle_z$.

Theorem

Assuming $f \in \mathcal{L}$, for any $h \in \mathbb{Z}$ (5) is finite and equals γ_h, and

$$[f]_+(z) = \sum_{h \geq 0} \langle z^{-h} f(z) \rangle_z \, z^h \qquad [f]_-(z) = \sum_{h < 0} \langle z^{-h} f(z) \rangle_z \, z^h.$$

The Case of a Single Variable

Causality, Anti-causality, and Invertibility

- For the class $\mathcal{L}$ we want to define notions of *causality*, *anti-causality*, and *invertibility*.
- From Theorem 1, causality can be defined as $[f]_- \equiv 0$.

Theorem

Assume $f \in \mathcal{L}$. *Then* $[f]_- \equiv 0$ *if and only if* $P(f) \subset \mathbb{C} \setminus \overline{U}$.

Definition

A function $f \in \mathcal{L}$ is causal if and only if $P(f) \subset \mathbb{C} \setminus \overline{U}$.

- Trivially, any entire function is causal, because it has no poles.
- Polynomials are an example of entire functions.

The Case of a Single Variable

Causality, Anti-causality, and Invertibility

- Analogously, we want to define the concept of invertibility by applying all these ideas to $1/f$.
- In other words, declare that f is invertible if and only if $1/f$ is causal. This prompts the following definition.

Definition

A function $f \in \mathcal{L}$ is invertible if and only if $Z(f) \subset \mathbb{C} \setminus \overline{U}$.

The Case of a Single Variable

Causality, Anti-causality, and Invertibility

- Finally, anti-causality occurs when $f(z^{-1})$ is a power series, and it follows from the proof of Theorem 2 that this occurs if and only if $[f]_+(z) \propto 1$. (It is equal to the zero coefficient $\langle f(z) \rangle_z$.)
- This in turn is equivalent to the poles of the function lying entirely inside the unit circle.

Definition

A function $f \in \mathcal{L}$ is anti-causal if and only if $P(f) \subset U$.

Cepstral Approximation

Defining the Cepstrum

- Suppose $f \in \mathcal{L}$ and is non-zero in some other annulus B containing the unit circle.
- Then f and $1/f$ are analytic on $\Omega = A \cap B$, and hence by 13.11(h) of Rudin (1987) there exists g analytic on Ω such that $f = \exp g$.
- We call g the logarithm of f, and in spectral analysis it is referred to as the *cepstrum*.
- g is analytic on Ω as well, and hence has a Laurent series.

Cepstral Approximation

Cepstrum Decomposition

- We can break g up additively into the past, present, and future portions.
- Letting $\xi_h = \langle z^{-h} g(z) \rangle_z$, we obtain

$$g(z) = \sum_{h \in \mathbb{Z}} \xi_h z^h = \sum_{h \leq -1} \xi_h z^h + \xi_0 + \sum_{h \geq 1} \xi_h z^h.$$

- This suggests defining new functions

$$f^+(z) = \exp\{ \sum_{h \geq 1} \xi_h z^h \}$$

$$f^-(z) = \exp\{ \sum_{h \geq 1} \xi_{-h} z^h \}.$$

Cepstral Approximation

Cepstrum Decomposition

- From these definitions follows the relationship

$$f(z) = f^{-}(z^{-1})\, e^{\xi_0}\, f^{+}(z). \tag{7}$$

- This is a multiplicative decomposition in terms of causal and anti-causal power series.

Cepstral Approximation

Cepstrum Properties

- $\log f^+(z)$ is a power series with radius of convergence given by the outer radius of Ω, and hence has no poles inside the unit circle (and therefore is causal).
- Hence $f^+(z)$ is also causal, and has a power series representation, whose coefficients can be explicitly computed in terms of $\{\xi_h\}_{h \geq 1}$ by the recursions of Pourahmadi (1984).
- Likewise, $f^-(z)$ is a causal power series with radius of convergence given by the reciprocal of the inner radius of Ω.
- $f^+(z)$ is called the causal factor, and $f^-(z^{-1})$ the anti-causal factor.

Cepstral Approximation

Calculation

- How do we calculate ξ_h in practice?
- A complex number $f(z)$ has polar form $r(z)e^{i\omega(z)}$, and $\log f(z) = \log r(z) + i\omega(z)$.
- Setting $g(z) = \log f(z)$, we have $\mathrm{Re}[g(z)] = \log r(z)$ and $\mathrm{Im}[g(z)] = \omega(z)$.
- With $\mathrm{Arg} f(z)$ computed by $\tan^{-1}(\mathrm{Im}[f(z)]/\mathrm{Re}[f(z)])$,

$$\xi_h = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{i\lambda h}\log|f(e^{-i\lambda})|\,d\lambda + \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{i\lambda h}\mathrm{Arg} f(e^{-i\lambda})\,d\lambda.$$

Bispectral Factorization

Defining the Bi-Cepstrum

- We now focus on the $k = 2$ case of the acgf.
- The relation between auto-cumulant and polyspectral density when $k = 2$ is given by

$$\gamma_{h_1,h_2} = \langle \langle z_1^{-h_1} z_2^{-h_2} f(z_1, z_2) \rangle_{z_1} \rangle_{z_2}. \tag{8}$$

- Assume there exists a constant $C > 0$ and rates $r_1, r_2 \in (0, 1)$ such that

$$|\gamma_{h_1,h_2}| \leq C r_1^{|h_1|} r_2^{|h_2|} \tag{9}$$

for all $h_1, h_2 \in \mathbb{Z}$.

Bispectral Factorization

Defining the Bi-Cepstrum

- Assuming $f(z_1, z_2)$ has double-Laurent coefficients satisfying (9), and is also non-zero in some annulus containing the unit torus, the bi-cepstrum exists.
- The bi-cepstral coefficients are

$$\xi_{h_1,h_2} = \langle \langle z_1^{-h_1} z_2^{-h_2} \left(\log |f(z_1, z_2)| + \mathrm{Arg} f(z_1, z_2) \right) \rangle_{z_1} \rangle_{z_2}.$$

- They also satisfy (9).

Bispectral Factorization

Symmetries of the Bispectrum

- Consider the symmetries of the third auto-cumulant function:

$$\begin{aligned}\gamma_{h_1,h_2} &= \gamma_{h_2,h_1} = \gamma_{-h_1,h_2-h_1} \\ &= \gamma_{-h_2,h_1-h_2} = \gamma_{h_2-h_1,-h_1} = \gamma_{h_1-h_2,-h_2}.\end{aligned}$$

- From these symmetries, we have corresponding symmetries in the bispectrum:

$$\begin{aligned}f(z_1,z_2) &= f(z_2,z_1) = f(z_1^{-1}z_2^{-1},z_2) \\ &= f(z_1,z_1^{-1}z_2^{-1}) = f(z_2,z_1^{-1}z_2^{-1}) = f(z_1^{-1}z_2^{-1},z_1).\end{aligned}$$

Bispectral Factorization

Symmetries of the Bi-cepstrum

- The bi-cepstrum must have the same six symmetries.
- This means that, for any $h_1, h_2 \in \mathbb{Z}$, the following multinomials have the same cepstral coefficient:

$$z_1^{h_1} z_2^{h_2},\ z_2^{h_1} z_1^{h_2},\ (z_1 z_2)^{-h_1} z_2^{h_2},$$
$$z_1^{h_1} (z_1 z_2)^{-h_2},\ z_2^{h_1} (z_1 z_2)^{-h_2},\ (z_1 z_2)^{-h_1} z_1^{h_2}.$$

- We will group terms, and denote the coefficients by τ_{h_1,h_2}.

Bispectral Factorization

Formulation of the Bi-cepstrum

- This means we can rewrite the bi-cepstrum as

$$g(z_1, z_2) = \sum_{(h_1,h_2)\in R} \tau_{h_1,h_2} \Big(z_1^{h_1} z_2^{h_2} + z_1^{h_2} z_2^{h_1} + (z_1 z_2)^{-h_1} z_2^{h_2} + z_2^{h_1} (z_1 z_2)^{-h_2} + z_1^{h_2} (z_1 z_2)^{-h_1} + z_1^{h_1} (z_1 z_2)^{-h_2} \Big),$$

where R is the tetrahedral cone

$$R = \{h_1, h_2 \in \mathbb{Z} : h_1 \geq h_2 \geq 0\}.$$

Bispectral Factorization

Formulation of the Bi-cepstrum

- By change of variable, this can be rewritten as

$$g(z_1, z_2) = \sum_{h_1, h_2 \geq 0} \tau_{h_1+h_2, h_2} \Big(z_1^{h_1} (z_1 z_2)^{h_2} + z_2^{h_1} (z_1 z_2)^{h_2} + (z_1^{-1} z_2^{-1})^{h_1} z_2^{-h_2} + z_2^{h_1} z_1^{-h_2} + (z_1^{-1} z_2^{-1})^{h_1} z_1^{-h_2} + z_1^{h_1} z_2^{-h_2} \Big).$$

- There are three axes of symmetry dividing $\mathbb{Z}^2$ into six regions, one of which is R. There are 4 triangular *cones*, as well as quadrants II and IV.
- There are six boundary *rays*, as well as the origin.

Bispectral Factorization

Formulation of the Bi-cepstrum

- How is τ related to cepstral ξ?

$$\tau_{k_1,k_2} = \begin{cases} \xi_{k_1,k_2} & \text{on interior regions} \\ \frac{1}{2}\xi_{k_1,k_2} & \text{on boundaries of regions} \\ \frac{1}{6}\xi_{0,0} & \text{at the origin.} \end{cases}$$

Bispectral Factorization

The Double Power Series

- Let D be a 2×2 matrix of ones in its upper triangle.
- Define the double-power series

$$G(u_1, u_2) = \sum_{\underline{h} > 0} \tau_{D\underline{h}} u_1^{h_1} u_2^{h_2},$$

where $\underline{h} > 0$ means not both h_1 and h_2 equal zero, but both are non-negative.

Bispectral Factorization

Decomposing the Bi-cepstrum

- Then the bi-cepstrum is compactly expressed as

$$g(z_1, z_2) = 6\tau_{0,0} + G(z_1, z_1 z_2) + G(z_2, z_1 z_2) + G(z_1^{-1} z_2^{-1}, z_1^{-1}) + G(z_1^{-1} z_2^{-1}, z_2^{-1}) + G(z_2, z_1^{-1}) + G(z_1, z_2^{-1}).$$

- Set $\Psi(z_1, z_2) = \exp G(z_1, z_2)$, and note $\Psi(0, 0) = 1$.

Bispectral Factorization

Decomposing the Bispectrum

- The factorization result is:

$$f(z_1, z_2) = e^{\xi_{0,0}} \Psi(z_1, z_1 z_2)\, \Psi(z_2, z_1 z_2)\, \Psi(z_1^{-1} z_2^{-1}, z_1^{-1}) \\ \Psi(z_1^{-1} z_2^{-1}, z_2^{-1})\, \Psi(z_2, z_1^{-1})\, \Psi(z_1, z_2^{-1}). \tag{10}$$

- This is the generalization of $f(z_1) = e^{\xi_0} \Psi(z_1) \Psi(z_1^{-1})$ for the spectral density.
- Every bispectrum satisfying (9) that is non-zero on an annulus about the unit torus has a bi-cepstrum satisfying (10), and conversely.

Trispectral Factorization

Defining the Tri-Cepstrum

- We now focus on the $k = 3$ case of the acgf.
- The relation between auto-cumulant and polyspectral density when $k = 3$ is given by

$$\gamma_{h_1,h_2,h_3} = \langle \langle \langle z_1^{-h_1} z_2^{-h_2} z_3^{-h_3} f(z_1, z_2, z_3) \rangle_{z_1} \rangle_{z_2} \rangle_{z_3}. \tag{11}$$

- Assume there exists a constant $C > 0$ and rates $r_1, r_2, r_3 \in (0, 1)$ such that

$$|\gamma_{h_1,h_2,h_3}| \leq C r_1^{|h_1|} r_2^{|h_2|} r_3^{|h_3|} \tag{12}$$

for all $h_1, h_2, h_3 \in \mathbb{Z}$.

Trispectral Factorization

Defining the Tri-Cepstrum

- Assuming $f(z_1, z_2, z_3)$ has tripe-Laurent coefficients satisfying (12), and is also non-zero in some annulus containing the unit 3-torus, the tri-cepstrum exists.
- The tri-cepstral coefficients are ξ_{h_1,h_2,h_3} equal

$$\langle \langle \langle z_1^{-h_1} z_2^{-h_2} z_3^{-h_3} \left(\log |f(z_1, z_2, z_3)| + \mathrm{Arg} f(z_1, z_2, z_3) \right) \rangle_{z_1} \rangle_{z_2} \rangle_{z_3}.$$

- They also satisfy (12).

Trispectral Factorization

Symmetries of the Trispectrum

- Consider the symmetries of the fourth auto-cumulant function:

$$\gamma_{h_1,h_2,h_3} = \gamma_{h_2,h_1,h_3} = \gamma_{h_1,h_3,h_2} = \gamma_{h_3,h_2,h_1} = \gamma_{h_3,h_1,h_2} = \gamma_{h_2,h_3,h_1}$$

$$\begin{aligned}\gamma_{h_1-h_3,h_2-h_3,-h_3} &= \gamma_{h_2-h_3,h_1-h_3,-h_3} = \gamma_{h_1-h_2,h_3-h_2,-h_2} \\ &= \gamma_{h_3-h_1,h_2-h_1,-h_1} = \gamma_{h_3-h_2,h_1-h_2,-h_2} = \gamma_{h_2-h_1,h_3-h_1,-h_1}\end{aligned}$$

$$\begin{aligned}\gamma_{h_1-h_3,-h_3,h_2-h_3} &= \gamma_{h_2-h_3,-h_3,h_1-h_3} = \gamma_{h_1-h_2,-h_2,h_3-h_2} \\ &= \gamma_{h_3-h_1,-h_1,h_2-h_1} = \gamma_{h_3-h_2,-h_2,h_1-h_2} = \gamma_{h_2-h_1,-h_1,h_3-h_1}\end{aligned}$$

$$\begin{aligned}\gamma_{-h_3,h_1-h_3,h_2-h_3} &= \gamma_{-h_3,h_2-h_3,h_1-h_3} = \gamma_{-h_2,h_1-h_2,h_3-h_2} \\ &= \gamma_{-h_1,h_3-h_1,h_2-h_1} = \gamma_{-h_2,h_3-h_2,h_1-h_2} = \gamma_{-h_1,h_2-h_1,h_3-h_1}.\end{aligned}$$

Trispectral Factorization

Symmetries of the Trispectrum

- From these symmetries, we have corresponding symmetries in the trispectrum:

$$
\begin{aligned}
& f(z_1, z_2, z_3) = f(z_2, z_1, z_3) = f(z_1, z_3, z_2) = f(z_3, z_2, z_1) = f(z_3, z_1, z_2) = f(z_2, z_3, z_1) \\
&= f(z_1, z_2, z_1^{-1} z_2^{-1} z_3^{-1}) = f(z_2, z_1, z_1^{-1} z_2^{-1} z_3^{-1}) = f(z_1, z_1^{-1} z_2^{-1} z_3^{-1}, z_2) \\
&\quad = f(z_1^{-1} z_2^{-1} z_3^{-1}, z_2, z_1) = f(z_1^{-1} z_2^{-1} z_3^{-1}, z_1, z_2) = f(z_2, z_1^{-1} z_2^{-1} z_3^{-1}, z_1) \\
&= f(z_1, z_1^{-1} z_2^{-1} z_3^{-1}, z_3) = f(z_1^{-1} z_2^{-1} z_3^{-1}, z_1, z_3) = f(z_1, z_3, z_1^{-1} z_2^{-1} z_3^{-1}) \\
&\quad = f(z_3, z_1^{-1} z_2^{-1} z_3^{-1}, z_1) = f(z_3, z_1, z_1^{-1} z_2^{-1} z_3^{-1}) = f(z_1^{-1} z_2^{-1} z_3^{-1}, z_3, z_1) \\
&= f(z_1^{-1} z_2^{-1} z_3^{-1}, z_2, z_3) = f(z_2, z_1^{-1} z_2^{-1} z_3^{-1}, z_3) = f(z_1^{-1} z_2^{-1} z_3^{-1}, z_3, z_2) \\
&\quad = f(z_3, z_2, z_1^{-1} z_2^{-1} z_3^{-1}) = f(z_3, z_1^{-1} z_2^{-1} z_3^{-1}, z_2) = f(z_2, z_3, z_1^{-1} z_2^{-1} z_3^{-1})
\end{aligned}
$$

Trispectral Factorization

Symmetries of the Tri-cepstrum

- The tri-cepstrum must have the same 24 symmetries.
- This means that, for any $h_1, h_2, h_3 \in \mathbb{Z}$, the following multinomials have the same cepstral coefficient:

$$
\begin{gathered}
z_1^{h_1} z_2^{h_2} z_3^{h_3},\ z_2^{h_1} z_1^{h_2} z_3^{h_3},\ z_1^{h_1} z_3^{h_2} z_2^{h_3},\ z_3^{h_1} z_2^{h_2} z_1^{h_3},\ z_3^{h_1} z_1^{h_2} z_2^{h_3},\ z_2^{h_1} z_3^{h_2} z_1^{h_3}, \\
z_1^{h_1} z_2^{h_2} (z_1 z_2 z_3)^{-h_3},\ z_2^{h_1} z_1^{h_2} (z_1 z_2 z_3)^{-h_3},\ z_1^{h_1} (z_1 z_2 z_3)^{-h_2} z_2^{h_3}, \\
(z_1 z_2 z_3)^{-h_1} z_2^{h_2} z_1^{h_3},\ (z_1 z_2 z_3)^{-h_1} z_1^{h_2} z_2^{h_3},\ z_2^{h_1} (z_1 z_2 z_3)^{-h_2} z_1^{h_3}, \\
z_1^{h_1} (z_1 z_2 z_3)^{-h_2} z_3^{h_3},\ (z_1 z_2 z_3)^{-h_1} z_1^{h_2} z_3^{h_3},\ z_1^{h_1} z_3^{h_2} (z_1 z_2 z_3)^{-h_3}, \\
z_3^{h_1} (z_1 z_2 z_3)^{-h_2} z_1^{h_3},\ z_3^{h_1} z_1^{h_2} (z_1 z_2 z_3)^{-h_3},\ (z_1 z_2 z_3)^{-h_1} z_3^{h_2} z_1^{h_3}, \\
(z_1 z_2 z_3)^{-h_1} z_2^{h_2} z_3^{h_3},\ z_2^{h_1} (z_1 z_2 z_3)^{-h_2} z_3^{h_3},\ (z_1 z_2 z_3)^{-h_1} z_3^{h_2} z_2^{h_3}, \\
z_3^{h_1} z_2^{h_2} (z_1 z_2 z_3)^{-h_3},\ z_3^{h_1} (z_1 z_2 z_3)^{-h_2} z_2^{h_3},\ z_2^{h_1} z_3^{h_2} (z_1 z_2 z_3)^{-h_3}.
\end{gathered} \tag{13}
$$

- We will group terms, and denote the coefficients by τ_{h_1,h_2,h_3}.

Trispectral Factorization

Formulation of the Tri-cepstrum

- Let R be the tetrahedral cone

$$R = \{h_1, h_2, h_3 \in \mathbb{Z} : h_1 \geq h_2 \geq h_3 \geq 0\}.$$

- There are six planes of symmetry dividing $\mathbb{Z}^3$ into 24 regions, one of which is R.
- There are several boundary *sheets*, as well as the origin.
- Let D be a 3×3 matrix of ones in its upper triangle.

Trispectral Factorization

Formulation of the Tri-cepstrum

- By change of variable, the tri-cepstrum can be rewritten as

$$g(z_1, z_2, z_3) = \sum_{\underline{h} \geq 0} \tau_{D\underline{h}} \cdot$$

$$\Big({z_1}^{h_1} (z_1 z_2)^{h_2} (z_1 z_2 z_3)^{h_3} + {z_2}^{h_1} (z_1 z_2)^{h_2} (z_1 z_2 z_3)^{h_3} + {z_1}^{h_1} (z_1 z_3)^{h_2} (z_1 z_2 z_3)^{h_3}$$
$${z_3}^{h_1} (z_2 z_3)^{h_2} (z_1 z_2 z_3)^{h_3} + {z_2}^{h_1} (z_2 z_3)^{h_2} (z_1 z_2 z_3)^{h_3} + {z_3}^{h_1} (z_1 z_3)^{h_2} (z_1 z_2 z_3)^{h_3}$$
$${z_1}^{h_1} (z_1 z_2)^{h_2} {z_3}^{-h_3} + {z_2}^{h_1} (z_1 z_2)^{h_2} {z_3}^{-h_3} + {z_1}^{h_1} (z_1 z_3)^{h_2} {z_2}^{-h_3}$$
$${z_3}^{h_1} (z_2 z_3)^{h_2} {z_1}^{-h_3} + {z_2}^{h_1} (z_2 z_3)^{h_2} {z_1}^{-h_3} + {z_3}^{h_1} (z_1 z_3)^{h_2} {z_2}^{-h_3}$$
$${z_1}^{h_1} (z_2 z_3)^{-h_2} {z_2}^{h_3} + {z_1}^{h_1} (z_2 z_3)^{-h_2} {z_2}^{-h_3} + {z_2}^{h_1} (z_1 z_3)^{-h_2} {z_1}^{-h_3}$$
$$(z_1 z_2 z_3)^{-h_1} (z_1 z_3)^{-h_2} {z_1}^{-h_3} + (z_1 z_2 z_3)^{-h_1} (z_1 z_2)^{-h_2} {z_1}^{-h_3} + {z_3}^{h_1} (z_1 z_2)^{-h_2} {z_1}^{-h_3}$$
$$(z_1 z_2 z_3)^{-h_1} (z_1 z_3)^{-h_2} {z_3}^{-h_3} + {z_2}^{h_1} (z_1 z_3)^{-h_2} {z_3}^{-h_3} + (z_1 z_2 z_3)^{-h_1} (z_2 z_3)^{-h_2} {z_3}^{-h_3}$$
$$(z_1 z_2 z_3)^{-h_1} (z_1 z_2)^{-h_2} {z_2}^{-h_3} + {z_3}^{h_1} (z_1 z_2)^{-h_2} {z_2}^{-h_3} + (z_1 z_2 z_3)^{-h_1} (z_2 z_3)^{-h_2} {z_2}^{-h_3} \Big) .$$

Trispectral Factorization

The Triple Power Series

- Define the triple-power series

$$G(u_1, u_2, u_3) = \sum_{\underline{h} > 0} \xi_{D\underline{h}} u_1^{h_1} u_2^{h_2} u_3^{h_3},$$

  where $\underline{h} > 0$ means not all of h_1, h_2, and h_3 equal zero, but all are non-negative.
- Set $\Psi(z_1, z_2, z_3) = \exp G(z_1, z_2, z_3)$, and note $\Psi(0, 0, 0) = 1$.

Trispectral Factorization

Decomposing the Tri-cepstrum

- Then the tri-cepstrum is compactly expressed as

$$
\begin{aligned}
g(z_1, z_2, z_3) &= 24\tau_{0,0,0} \\
&+ G(z_1, z_1 z_2, z_1 z_2 z_3) + G(z_2, z_1 z_2, z_1 z_2 z_3) + G(z_1, z_1 z_3, z_1 z_2 z_3) \\
&+ G(z_3, z_2 z_3, z_1 z_2 z_3) + G(z_2, z_2 z_3, z_1 z_2 z_3) + G(z_3, z_1 z_3, z_1 z_2 z_3) \\
&+ G(z_1, z_1 z_2, z_3^{-1}) + G(z_2, z_1 z_2, z_3^{-1}) + G(z_1, z_1 z_3, z_2^{-1}) \\
&+ G(z_3, z_2 z_3, z_1^{-1}) + G(z_2, z_2 z_3, z_1^{-1}) + G(z_3, z_1 z_3, z_2^{-1}) \\
&+ G(z_1, z_2^{-1} z_3^{-1}, z_2) + G(z_1, z_2^{-1} z_3^{-1}, z_2^{-1}) + G(z_2, z_1^{-1} z_3^{-1}, z_1^{-1}) \\
&+ G(z_1^{-1} z_2^{-1} z_3^{-1}, z_1^{-1} z_3^{-1}, z_1^{-1}) + G(z_1^{-1} z_2^{-1} z_3^{-1}, z_1^{-1} z_2^{-1}, z_1^{-1}) + G(z_3, z_1^{-1} z_2^{-1}, z_1^{-1}) \\
&+ G(z_1^{-1} z_2^{-1} z_3^{-1}, z_1^{-1} z_3^{-1}, z_3^{-1}) + G(z_2, z_1^{-1} z_3^{-1}, z_3^{-1}) + G(z_1^{-1} z_2^{-1} z_3^{-1}, z_2^{-1} z_3^{-1}, z_3^{-1}) \\
&+ G(z_1^{-1} z_2^{-1} z_3^{-1}, z_1^{-1} z_2^{-1}, z_2^{-1}) + G(z_3, z_1^{-1} z_2^{-1}, z_2^{-1}) + G(z_1^{-1} z_2^{-1} z_3^{-1}, z_2^{-1} z_3^{-1}, z_2^{-1}).
\end{aligned}
$$

Trispectral Factorization

Decomposing the Trispectrum

- The factorization result is:

$$
\begin{aligned}
& f(z_1, z_2, z_3) = e^{\xi_{0,0,0}} \\
& \Psi(z_1, z_1 z_2, z_1 z_2 z_3)\Psi(z_2, z_1 z_2, z_1 z_2 z_3)\Psi(z_1, z_1 z_3, z_1 z_2 z_3) \\
& \Psi(z_3, z_2 z_3, z_1 z_2 z_3)\Psi(z_2, z_2 z_3, z_1 z_2 z_3)\Psi(z_3, z_1 z_3, z_1 z_2 z_3) \\
& \Psi(z_1, z_1 z_2, z_3^{-1})\Psi(z_2, z_1 z_2, z_3^{-1})\Psi(z_1, z_1 z_3, z_2^{-1}) \\
& \Psi(z_3, z_2 z_3, z_1^{-1})\Psi(z_2, z_2 z_3, z_1^{-1})\Psi(z_3, z_1 z_3, z_2^{-1}) \\
& \Psi(z_1, z_2^{-1} z_3^{-1}, z_2)\Psi(z_1, z_2^{-1} z_3^{-1}, z_2^{-1})\Psi(z_2, z_1^{-1} z_3^{-1}, z_1^{-1}) \\
& \Psi(z_1^{-1} z_2^{-1} z_3^{-1}, z_1^{-1} z_3^{-1}, z_1^{-1})\Psi(z_1^{-1} z_2^{-1} z_3^{-1}, z_1^{-1} z_2^{-1}, z_1^{-1})\Psi(z_3, z_1^{-1} z_2^{-1}, z_1^{-1}) \\
& \Psi(z_1^{-1} z_2^{-1} z_3^{-1}, z_1^{-1} z_3^{-1}, z_3^{-1})\Psi(z_2, z_1^{-1} z_3^{-1}, z_3^{-1})\Psi(z_1^{-1} z_2^{-1} z_3^{-1}, z_2^{-1} z_3^{-1}, z_3^{-1}) \\
& \Psi(z_1^{-1} z_2^{-1} z_3^{-1}, z_1^{-1} z_2^{-1}, z_2^{-1})\Psi(z_3, z_1^{-1} z_2^{-1}, z_2^{-1})\Psi(z_1^{-1} z_2^{-1} z_3^{-1}, z_2^{-1} z_3^{-1}, z_2^{-1}).
\end{aligned}
$$

Polyspectral Factorization

Describing the Tetrahedral Cone

- We shall define τ supported on tetrahedral cone:

$$R = \{h_1, h_2, \ldots, h_k \in \mathbb{Z} : h_1 \geq h_2 \geq \ldots \geq h_k \geq 0\}.$$

- Let D denote an aggregation matrix such that

$$D\,\underline{h} = \begin{bmatrix} 1 & 1 & \ldots & 1 \\ 0 & 1 & \ldots & 1 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & \ldots 0 & 1 & \end{bmatrix} \begin{bmatrix} h_1 \\ h_2 \\ \vdots \\ h_k \end{bmatrix} = \begin{bmatrix} h_1 + h_2 + \ldots + h_k \\ h_2 + \ldots + h_k \\ \vdots \\ h_k \end{bmatrix}.$$

- Define the notation $\underline{z}^{\underline{j}}$ as a shorthand for $\prod_{\ell=1}^{k} z_\ell^{j_\ell}$.

Polyspectral Factorization

Describing the Autocumulant Symmetries

- Adopt the notation and concepts of Berg (2008).
- Consider σ to be any permutation in the group $\mathcal{S}_{k+1}$ of all permutations on $k+1$ elements; there are $(k+1)!$ such permutations.
- The symmetries of the autocumulant function can be viewed as including the index 0 to the k lag indices, and considering any permuting via σ of these $k+1$ numbers.
- There exists a *group representation* ρ, which maps the action of these permutations to a $k \times k$ matrix $\rho(\sigma)$, which maps $\underline{h}$ to $\rho(\sigma)\underline{h}$, the new set of indices that indicate autocumulant symmetry, i.e., $\gamma_{\underline{h}} = \gamma_{\rho(\sigma)\underline{h}}$.

Polyspectral Factorization

Describing the Polyspectral Symmetries

- We can also describe the symmetries of polyspectra using the group representation notation.

Theorem

For any $\sigma \in \mathcal{S}_{k+1}$, the value of the polyspectra at variables z_m for $1 \leq m \leq k$ is the same if z_m is replaced by

$$\prod_{j=1}^{k} z_j^{[\rho(\sigma^{-1})]_{jm}}.$$

Polyspectral Factorization

Defining τ

- For any $\underline{\ell} \in \mathbb{Z}^k$, for each $\sigma \in \mathcal{S}_{k+1}$ we can find $\underline{h} = D^{-1} \rho(\sigma^{-1}) \underline{\ell}$.
- Allowing for some boundary cases, we can define τ via

$$\sum_{\underline{\ell} \in \mathbb{Z}^k} \xi_{\underline{\ell}} \, \underline{z}^{\underline{\ell}} = \sum_{\underline{h} \geq 0} \tau_{D\underline{h}} \sum_{\sigma \in \mathcal{S}_{k+1}} \underline{z}^{\rho(\sigma) D \underline{h}}. \tag{14}$$

Polyspectral Factorization

Relating ξ to τ

- By integrating both sides against $\underline{z}^{-\underline{\ell}}$, we obtain

$$\xi_{\underline{\ell}} = \sum_{\sigma \in \mathcal{S}_{k+1}} \tau_{\rho(\sigma^{-1})\underline{\ell}}. \tag{15}$$

- We can compute τ as follows: take $\underline{\ell} \in R$, and determine how many $\sigma \in \mathcal{S}_{k+1}$ exist such that $\rho(\sigma^{-1})\underline{\ell} \in R$ and yield the same value $\underline{\ell}$.
- The number of such permutations depends on where $\underline{\ell}$ lies, and what symmetries it is invariant under; in general, if it is subject to m out of k of the boundary constraints of ∂R, then it is invariant with respect to $m!$ permutations.
- Then $\tau_{D\underline{h}} = \xi_{\underline{\ell}}/m!$

Polyspectral Factorization

Decomposing the Poly-Cepstrum

- Rewrite (14) as

$$\sum_{\underline{\ell} \in \mathbb{Z}^k} \xi_{\underline{\ell}}\, \underline{z}^{\underline{\ell}} = k! \tau_{\underline{0}} + \sum_{\sigma \in \mathcal{S}_{k+1}} \sum_{\underline{h} > 0} \tau_{D\underline{h}} \prod_{m=1}^{k} \left(\prod_{j=1}^{k} z_j^{[\rho(\sigma) D]_{jm}} \right)^{h_m}.$$

- Because of the structure of D and $\rho(\sigma)$, the entries of $\rho(\sigma)D$ are either 0, 1, or -1.
- Hence the power $[\rho(\sigma)D]_{jm}$ of z_j indicates either the variable is omitted, retained, or flipped.

Polyspectral Factorization

Defining G and Ψ_{k+1}

$$G(\underline{u}) = \sum_{\underline{h}>0} \tau_{D\underline{h}} \, \underline{u}^{\underline{h}},$$

and

$$\Psi_{k+1}(\underline{u}) = \exp\{G(\underline{u})\},$$

which satisfies $\Psi_{k+1}(\underline{0}) = 1$.

Polyspectral Factorization

Rewriting the Poly-Cepstrum

- Hence the poly-cepstrum is

$$g(\underline{z}) = \xi_{\underline{0}} + \sum_{\sigma \in \mathcal{S}_{k+1}} G\left(\left\{ \prod_{j=1}^{k} z_j^{[\rho(\sigma)D]_{jm}} \right\}_{m=1}^{k} \right). \tag{16}$$

- The polyspectral factorization is

$$f(\underline{z}) = e^{\xi_{\underline{0}}} \prod_{\sigma \in \mathcal{S}_{k+1}} \Psi_{k+1}\left(\left\{ \prod_{j=1}^{k} z_j^{[\rho(\sigma)D]_{jm}} \right\}_{m=1}^{k} \right). \tag{17}$$

Polyspectral Factorization

Obtaining ψ from τ

- The coefficient ψ can be written

$$\psi_{\underline{j}}^{(k+1)} = \frac{1}{(2\pi)^k} \int_{[-\pi,\pi]^k} \exp\{i\underline{j}'\underline{\lambda}\} \exp\{\sum_{\underline{h}>0} \tau_{D\underline{h}} \cos(\underline{h}'\underline{\lambda})\} \exp\{-i \sum_{\underline{h}>0} \tau_{D\underline{h}} \sin(\underline{h}'\underline{\lambda})\} \, d\underline{\lambda}.$$

Polyspectral Modeling

Nonlinear Time Series

- We can describe stationary nonlinear time series through their autocumulants γ.
- Equivalently through their polyspectra f.
- For certain class of f (geometric decay of γ, and no zeroes on thickened torus), the poly-cepstrum ξ exists.
- We obtain ψ via cepstral relation of power series.

$$\{\gamma\} \mapsto \{\xi\} \mapsto \{\tau\} \mapsto \{\psi\} \mapsto \{\gamma\}.$$

Polyspectral Modeling

Semi-parametric Approach

- The conditions on $\{\tau\}$ are simple: coefficients can be any $\mathbb{R}$.
- Consider $\underline{h} > 0$, up to some threshold m, and map to tetrahedral cone by $D\underline{h}$.
- Generate k-fold power series to define $G(\underline{z})$, and hence $\Psi(\underline{z})$.
- The polyspectral factorization result depends on how G functions are multiplied to obtain the polyspectrum.
- Resulting $\{\gamma\}$ has all constraints (which generalize pd condition) automatically enforced.

Polyspectral Modeling

Applications and Future Work

1. Modeling: treat $\{\tau\}$ as parameters, and compare resulting f to empirical estimates of the polyspectra.
2. Entropy Philosophy: try to *transform* a time series such that $\tau \equiv 0$, rendering polyspectra to be constant (a higher order white noise).
3. Filtering: can we filter with $\Psi(\underline{z})^{-1}$ to get white noise? By Shiryaev (1960), this is not true.
4. Residuals: after transforming, assess residuals through τ
5. Simulation: can we filter a white noise with $\Psi(\underline{z})$ to get a given process? Again by Shiryaev (1960), this is not so simple.

Contact

- tucker.s.mcelroy@census.gov

Ahlfors, L. V. 1979. *Complex Analysis*. McGraw-Hill.

Berg, Arthur. 2008. "Multivariate Lag-Windows and Group Representations." *Journal of Multivariate Analysis* 99 (10): 2479–96.

Brillinger, David R. 1981. *Time Series: Data Analysis and Theory*. Vol. 36. Siam.

McElroy, Tucker S, and Dimitris N Politis. 2020. *Time Series: A First Course with Bootstrap Starter*. CRC Press.

Pourahmadi, Mohsen. 1984. "Taylor Expansion of and Some Applications." *The American Mathematical Monthly* 91 (5): 303–7.

Rudin, W. 1987. *Real and Complex Analysis*. McGraw-Hill.

Shiryaev, Albert Nikolaevich. 1960. "Some Problems in the Spectral Theory of Higher-Order Moments. I." *Theory of Probability & Its Applications* 5 (3): 265–84.