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Tucker McElroy[a]; Thomas M. Trimbur[b]
[a] U.S. Census Bureau, Washington, District of Columbia, USA [b] Federal Reserve Board, Washington, District of Columbia, USA

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ON THE DISCRETIZATION OF CONTINUOUS-TIME FILTERS FOR NONSTATIONARY STOCK AND FLOW TIME SERIES

Tucker McElroy[1] and Thomas M. Trimbur[2]

[1] *U.S. Census Bureau, Washington, District of Columbia, USA*
[2] *Federal Reserve Board, Washington, District of Columbia, USA*

□ *This article discusses the discretization of continuous-time filters for application to discrete time series sampled at any fixed frequency. In this approach, the filter is first set up directly in continuous-time; since the filter is expressed over a continuous range of lags, we also refer to them as continuous-lag filters. The second step is to discretize the filter itself. This approach applies to different problems in signal extraction, including trend or business cycle analysis, and the method allows for coherent design of discrete filters for observed data sampled as a stock or a flow, for nonstationary data with stochastic trend, and for different sampling frequencies. We derive explicit formulas for the mean squared error (MSE) optimal discretization filters. We also discuss the problem of optimal interpolation for nonstationary processes – namely, how to estimate the values of a process and its components at arbitrary times in-between the sampling times. A number of illustrations of discrete filter coefficient calculations are provided, including the local level model (LLM) trend filter, the smooth trend model (STM) trend filter, and the Band Pass (BP) filter. The essential methodology can be applied to other kinds of trend extraction problems. Finally, we provide an extended demonstration of the method on CPI flow data measured at monthly and annual sampling frequencies.*

Keywords Continuous time processes; Hodrick–Prescott filter; Interpolation; Linear filtering; Signal extraction.

JEL Classification C22; C51.

1. INTRODUCTION

In the analysis of time series, it is common to apply different filters to study the properties of interest. For instance, the well-known filter of

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Address correspondence to Thomas M. Trimbur, Federal Reserve Board, 20th Street and Constitution Avenue NW, Washington, DC 20551, USA; E-mail: thomas.m.trimbur@frb.gov

Hodrick and Prescott (HP) (1997) is widely used to remove the trend. The possibility of spurious results with the HP filter has been documented by Harvey and Jaeger (1993). In some cases, the HP filter may still give a reasonable detrended series, as for quarterly U.S. real GDP. For other series, it is sometimes possible to make a simple adjustment to the HP filter to give a plausible result; for instance, Ravn and Uhlig (2002) suggest how to adapt the HP filter for frequency of observation.

The model underlying the HP filter suffers from serious limitations; essentially, it includes no role for a cyclical component. Recent attention has turned to Band-Pass (BP) filters, which also cut out high frequency parts. For instance, Baxter and King (1999) presented a simple approximation to the "ideal" filter for finite samples. The gain function of a filter shows how the amplitudes of different frequency parts are affected when the filter is applied. The "ideal" filter has a block-like gain function that selects out a particular range of frequencies and completely annihilates frequencies outside this range. A model-based interpretation of the "ideal" filter is given in Harvey and Trimbur (2003); this solves the endpoint problem, but it also raises questions about using the "ideal" filter in the first place. Murray (2003) and Harvey and Trimbur (2003) discuss the possible distortions associated with the "ideal" filter. As with the HP filter, the technique seems to work relatively well for quarterly U.S. real GDP. Even in this case, however, the shape of the gain function may be misleading. More generally, the class of generalized Butterworth filters give a versatile class of gain functions that adapt to the given time series; the band-pass filter is more flexible to account for the kind of cyclical behavior that occurs in economics.

In this article our interest centers on consistent filtering across different series. In particular, we focus on consistency across stock and flow variables and across different sampling intervals. This question is crucial in many applications, for instance, in macroeconomics where the researcher investigates stylized facts for a set of series. Ravn and Uhlig (2002) adopt the HP filter and focus on the sampling interval, but they advise the same adjustment for stock and flow. This result constrasts with our derivations, which highlight specific differences in stock and flow series. Our general aim is to develop coherent signal extraction filters for econometrics, including a set of low-pass filters such as the HP filter and a set of band-pass filters. To our knowledge, this represents the first attempt to design consistent band-pass filters for stock and flow data with different sampling frequencies.

An elegant way to formulate the problem is to set up a continuous-time model. Harvey (1989) and Harvey and Stock (1993) develop unobserved components models in continuous-time; they proceed in the usual manner, by deriving the discrete-time model implied by the underlying model, that is, they *discretize* the model. The discretization depends on the sampling interval and on whether the series is stock or flow. Further, the discretization of the model induces correlation between components, even if the original components are uncorrelated. In this article, we suggest a more direct method where the discrete filters are immediately designed from the same source through formulas for stock and flow series.

One aim of Harvey and Stock (1993) is to allow the estimation of components between observation points, or *interpolation*; they do so by setting up a state space model at a finer interval. In our framework, in contrast, the interpolated components are directly estimated through formulas for the discrete filters. Our method starts by setting up filters in continuous-time, called continuous-lag filters because the lag operator now takes on any real power; that is, the lag/lead index of the filter coefficients can take on any real value. These filters generalize the usual discrete-lag filters, and McElroy and Trimbur (2006) prove an analogous result to Bell (1984) for this more general case. In particular, the continuous-lag filters are derived as the optimal estimator in the signal extraction problem formulated in continuous-time.

The next step is to derive the discrete filters implied by the continuous-lag filter and the underlying dynamics. In this article, we focus on this *discretization of the filter*. As the filters all stem from one source, the continuous-lag filter, they are coherent with one another. Furthermore, they are given by explicit formulas, so to find the implications of a different sampling frequency, one simply changes the value of δ in the expressions for the weights.

Bergstrom (1988) argues in favor of continuous-time models for underlying economic processes. For example, for many flow variables linked to the macroeconomy, increments from production and exchange occur on a more or less constant basis. The continuous-lag filter we derive is fundamental to the signal extraction model. Furthermore, it gives a unified basis for filtering economic data, which may be sampled as a stock or a flow, are often nonstationary with stochastic trend, and may differ in sampling frequency. A further advantage is that we can design continuous-lag filters to target some property of interest, for instance velocity and acceleration filters for turning point analysis. The discrete filters are optimal in the mean squared error (MSE) sense among linear estimators; this is in contrast to the Reproducing Kernel Hilbert Space methodology of Dagum and Bianconcini (2008), which does not incorporate information about the covariance structure of the observed process into the resulting filter weights.

Section 2 reviews continuous-time processes and filters, setting the background and notation. Section 3 discusses coherent discrete filters for stocks and flows and presents the main results: Theorems 1 and 2 construct the discrete filters for stock and flow data consistent with a general continuous-lag filter. These results represent the first derivation of general filters for any sampling interval δ, and they include interpolations. In Section 4, we give examples, starting with simple trend plus noise models; the smooth trend case shows how to generalize the HP filter so that it adapts to stocks and flows with changing δ and how to do interpolation. We also derive the discrete filters for the continuous-lag BP filter. Note that the method carries over to more general problems. Section 5 shows an application to the Consumer Price Index (CPI), and Section 6 concludes. Derivations are in the Appendices.

2. CONTINUOUS-LAG FILTERS

This section presents the theoretical framework for the analysis of filters in continuous-time. For background in this area, we refer to Hannan (1971) and Priestley (1981). Let $y(t)$ for $t \in \mathbb{R}$ denote a real-valued continuous-time process that is measurable at each t. The process is weakly stationary by definition if it has constant mean—set to zero for simplicity—and an autocovariance function $R_y(h) = \mathbb{E}[y(t)y(t+h)]$ that depends only on the lag $h \in \mathbb{R}$. So if $y(t)$ is a Gaussian process, R_y completely describes the dynamics. A convenient class of stationary continuous-time processes has the form $y(t) = (\theta * \epsilon)(t) = \int_{-\infty}^{\infty} \theta(x)\epsilon(t-x)\,dx$, where $\theta(\cdot)$ is square integrable on $\mathbb{R}$ and $\epsilon(t)$ is continuous-time white noise (CN). In this case, $R_y(h) = (\theta * \theta^-)(h)$, where $\theta^-(x) = \theta(-x)$ and $*$ is the convolution operator.

The power spectrum of a weakly stationary continuous-time process $y(t)$ is the Fourier Transform of its autocovariance function R_y:

$$f_y(\lambda) = \mathcal{F}[R_y](\lambda), \qquad \lambda \in \mathbb{R}. \tag{1}$$

When R_y is integrable, $f_y(\lambda)$ is guaranteed to exist, and the process $y(t)$ is called stochastically continuous. The inverse Fourier Transform of an integrable function g is defined by $\mathcal{F}^{-1}[g](x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} g(\lambda)e^{i\lambda x}\,d\lambda$ for $x \in \mathbb{R}$. We will also need to consider *generalized spectra*, which includes non-integrable functions f_y. For example, CN has generalized spectrum given by a constant—which is bounded but not integrable. In this case, (1) is interpreted as being true in the sense of tempered distributions (Folland, 1995).

Define the continuous-time lag operator L by $L^x y(t) = y(t-x)$ for any $x \in \mathbb{R}$ and for all times $t \in \mathbb{R}$. Denote the identity operator L^0 by 1, just as in discrete time. Then a *Continuous-Lag Filter* is an operator $\Psi(L)$ with associated *weighting kernel* ψ (an integrable symmetric function) whose effect on a process $y(t)$ is given by

$$\Psi(L)y(t) = \int_{-\infty}^{\infty} \psi(x)y(t-x)\,dx = (\psi * y)(t).$$

The kernel is of generic order in this definition. We can write $\Psi(L) = \int_{-\infty}^{\infty} \psi(x)\, L^x\, dx$ in analogy with discrete-lag filters. Now the discrete weights, ψ_j's for integer j, are replaced by a weight density function $\psi(x)$.

The frequency response function (frf) describes how various component frequencies are attenuated, amplified, and delayed by application of the filter. To compute this function, replace L in $\Psi(L)$ by the argument $e^{-i\lambda}$ to obtain $\Psi(e^{-i\lambda}) = \int_{-\infty}^{\infty} \psi(x) e^{-i\lambda x} dx$, for $\lambda \in \mathbb{R}$ so that $\Psi(e^{-i\lambda}) = \mathcal{F}[\psi]$. As in discrete time, passing an input process through the filter $\Psi(L)$ multiplies the spectrum by the squared modulus of $\Psi(e^{-i\lambda})$. Now, however, the domain of $\Psi(e^{-i\lambda})$ is the entire real line.

For nonstationary models, the derivative operator D plays an analogous role to the difference operator used in discrete-time. In the time domain, $D = -\log L$, and the frequency response is $i\lambda$. The operator D is used to define the continuous-time analogue of the standard Autoregressive Integrated Moving Average (ARIMA) class. That is, in the same way that ARIMA models are difference equations built on discrete-time white noise, CARIMA (continuous ARIMA) models are defined as stochastic differential equations built on continuous-time white noise. See Brockwell and Marquardt (2005) for further discussion.

Let $\mathscr{C}^d$ denote the space of all processes that are dth order stochastically differentiable. Given $y \in \mathscr{C}^d$, we say that y is integrated of order d, or $y(t) \sim I(d)$. Then $w(t) = D^d y(t)$ is a weakly stationary continuous-time process, but $D^0 y(t), \ldots, D^{d-1} y(t)$ are nonstationary. The CARIMA(p, d, q) class of processes is indexed by the order of integration. $\mathscr{C}^0$ refers to the space of stochastically continuous processes, for which an orthogonal increments representation exists by Theorem 4.11.1 of Priestley (1981).

This article focusses on deriving the set of discrete filters consistent with a continuous-lag filter $\Psi(L)$. In some cases, we might only know the order d of integration of the input process $y(t)$. Often, however, the filter $\Psi(L)$ may be derived from a signal extraction problem, such as

$$y(t) = s(t) + n(t), \tag{2}$$

where the decomposition has signal $s(t)$ and noise $n(t)$. Assume that $s(t) \sim I(d)$ but that $n(t)$ is weakly stationary—possibly $n \notin \mathscr{C}^0$. Let $D^d s(t) = u(t)$, and assume that $u(t)$ and $n(t)$ are uncorrelated with each other. We allow for a non-integrable spectrum for the differenced series, that is, $f_w(\lambda) = f_u(\lambda) + \lambda^{2d} f_n(\lambda)$, interpreted as a generalized spectrum. Model (2) just has two components, so it represents the simplest case of interest. We can equally well consider $n(t)$ as the target, for instance, if interest centers on detrending a series.

More generally, we can consider decomposition models formulated to describe the overall dynamics of the series, so the signal extraction filters are consistent with the properties of $y(t)$. We can design mutually consistent continuous-lag filters as low-pass and BP filters. McElroy and Trimbur (2006) derive the generalized Butterworth filters in continuous-time. The HP filter and the other Butterworth low-pass filters are defined as specific examples. Note that the class of continuous-lag BP filters includes the continuous-time version of the "ideal" filter as a special cases.

We prefer using filters designed for consistency with the input series, and we recognize the risk of using a nonparametric weighting kernel. Still, in some cases, a prespecified filter might prove adequate, and our framework handles a nonparametric approach equally well. Then the discrete set of filters are mutually consistent and consistent with the continuous-lag filter, and the consistency of the continuous-lag filter becomes a separate question. In this way, we could derive the filters corresponding to the continuous-time Henderson filter proposed by Dagum and Bianconcini (2008); their filter underlies the Henderson trend filters that are commonly used in seasonal adjustment methods such as X-12-ARIMA.

3. THE COHERENT SET OF DISCRETE FILTERS

In the hypothetical case of having continuous-time data $y(t)$, we could estimate the target signal by $x(t) = \Psi(L)y(t) = (\psi * y)(t)$, $\Psi(L)$ is a well-designed continuous-lag filter. In practice, of course, we just have discretely sampled data that can come in a variety of forms, and further, we may want information about different functions of the signal, for example, its average over a certain interval or its value or rate of change at some time point between observations. In this section, we discuss how to convert a given continuous-lag filter $\Psi(L)$ into discrete filters for each situation of interest. Our focus lies on exact optimal solutions, that is, the minimum MSE estimators in the linear class.

3.1. Stock and Flow

In economics, most time series are classified as either stock or flow, depending on how the data are sampled from an underlying process. Stock observations are estimates of levels at particular time points, while flow observations are accumulations over time intervals. These properties are formalized by setting up a continuous-time model with characteristic measurement equations; see, for example, Harvey (1989, Chapter 9).

Suppose that the observed data are measured at evenly spaced intervals of length δ. The basic timing unit is defined in the fundamental continuous-time setting, and is set so that $\delta = 1$ corresponds to these time units. For example, if we have annual units, then for quarterly data the sampling interval is $\delta = 1/4$ and the sampling frequency is 4 observations per year. In the analysis that follows, results are derived for general δ; the effects of changing δ are, therefore, explicit in the formulas. Observations in the discretized series are indexed by integer values τ, so that the τth observation occurs at the time $\delta\tau$. Given the sampling interval $\delta > 0$, a stock observation at the τth time point is defined as

$$y_\tau = y(\delta\tau). \tag{3}$$

We never make stock observations of processes that are not stochastically continuous, since this is not well-defined mathematically (as shown in Appendix A.1, the spectral density of the stock observations of such a process would be undefined). A series of flow observations has the form

$$y_\tau = \int_{\delta(\tau-1)}^{\delta\tau} y(v)\,dv. \tag{4}$$

Note that flow variables may also be formulated as differenced stock variables, if the stock variable can be expressed as the cumulative integral of some other underlying process.

3.2. Discretization of Filters

Now assuming that $y \in \mathscr{C}^d$ and that $D^d y(t) = w(t)$, where w is stationary, then from Hannan (1971, p. 81) it follows that

$$y(t) = \sum_{j=0}^{d-1} \frac{t^j}{j!} y^{(j)}(0) + [\mathcal{J}^d w](t), \tag{5}$$

where $y(0), y^{(1)}(0), \ldots, y^{(d-1)}(0)$ represent successive derivatives of $y(t)$ evaluated at time zero. The $\mathcal{J}$ operator is defined by $[\mathcal{J} w](t) = \int_0^t w(z)\,dz$, and $\mathcal{J}^d w$ is obtained inductively for $d > 1$. Using integration by parts, we can write $[\mathcal{J}^{d+1} w](t) = \int_0^t w(s)(t-s)^d\,ds/d!$, for any $t \in \mathbb{R}$. Now a stock observation of (5) at time $\delta\tau$ satisfies

$$y_\tau = \sum_{j=0}^{d-1} \frac{\delta^j y^{(j)}(0)}{j!} \tau^j + [\mathcal{J}^d w](\delta\tau). \tag{6}$$

The process $\{y_\tau\}$ is a d-integrated discrete-time process, and is reduced to stationarity by taking d differences (with $B = L^\delta$):

$$w_\tau = (1 - B)^d y_\tau = (1 - B)^d [\mathcal{J}^d w](\delta\tau). \tag{7}$$

The stationarity is proved in Appendix A.2; note that $w_\tau \neq w(\delta\tau)$. Now the process $\{y_\tau\}$ can be completely described through $\{w_\tau\}$ and d initial values, say $y^* = (y_0, y_{-1}, \ldots, y_{1-d})$ as shown in Bell (1984). Note that these initial values are distinct from the $\{y^{(j)}(0)\}$ when $d > 1$. We have two assumptions on the initial values:

$$y^* \text{ is uncorrelated with } \{w_\tau\}_{\tau=-\infty}^{\infty} \tag{8}$$

$$y^* \text{ is uncorrelated with } \{w(t)\}_{t\in\mathbb{R}}. \tag{9}$$

Clearly, (9) implies (8), since the former is concerned with the stochastic process $w(t)$ at all times t, whereas the weaker condition is only concerned with the (differenced) sampled values. While (8) is sufficient for a purely discrete-time setup (e.g., when no interpolation is considered) and is implied by Assumption A of Bell (1984)—a commonly employed assumption in the theory of signal extraction—the stronger condition (9) is needed to establish optimality in the more general case. If, on the other hand, we flow-observe (5), then

$$y_\tau = \sum_{j=0}^{d-1} \frac{\delta^{j+1} y^{(j)}(0)}{(j+1)!} \left(\tau^{j+1} - (\tau-1)^{j+1}\right) + \int_{\delta\tau-\delta}^{\delta\tau} [\mathcal{J}^d w](v)\, dv. \tag{10}$$

This too is a d-integrated discrete-time process, and in this case d differences yields:

$$w_\tau = (1-B)^d y_\tau = (1-B)^d \int_{\delta\tau-\delta}^{\delta\tau} [\mathcal{J}^d w](v)\, dv. \tag{11}$$

Now the initial values y^* and $\{w_\tau\}$ have different formulas, but we still refer to the initial value conditions (8) and (9), with w_τ interpreted appropriately.

Note that the appropriate range of frequencies corresponding to δ is $[-\pi/\delta, \pi/\delta]$. For any frequency outside this interval, the discrete-time behavior is equivalent to that of an "alias" frequency within the interval; see Koopmans (1974). Below we use the concept of the "fold" of a spectral density: $[f]_\delta(\lambda) = \delta^{-1} \sum_{l=-\infty}^{\infty} f(\lambda + 2\pi l/\delta)$. This gives the definition; if $\mathcal{F}^{-1}[f] = R$, it follows that $[f]_\delta(\lambda) = \sum_{h=-\infty}^{\infty} R(\delta h) e^{-i\lambda\delta h}$ as well, so $[f]_\delta$ is the spectral density of the stock-sampled (stationary) series (3). A discussion of the derivation and etymology of this concept is included in Appendix A.1 (also see Koopmans, 1974).

Now we address the problem of finding the minimal MSE *linear* estimate of $x(t)$, given the data $Y = \{y_\tau\}$, which is either stock- or flow-sampled according to (3) or (4), respectively. Here $t = \delta\tau + \delta c$, where τ is the greatest integer such that $\delta\tau \leq t$, and $c \in [0, 1)$. Thus, c determines to what extent $x(t)$ is placed in-between the sampling times; allowing for $c \neq 0$ in our optimal filters provides interpolation estimates of the underlying trend. The optimal solution can then be written as $\Psi_c(B)y_\tau$, where $\Psi_c(B)$ is a discrete-lag filter with each coefficient dependent upon c. Below, we present formulas for the frf $\Psi_c(e^{-i\lambda})$ (for $\lambda \in [-\pi/\delta, \pi/\delta]$) of the optimal filter, for both the stock and flow sampling cases, given that $y(t)$ is an integrated process of order d satisfying (9). In both the stock and flow cases, $\Psi_c(e^{-i\lambda})$ will be seen to depend on c, d, and δ, on the continuous time frf $\mathcal{F}[\psi] = g$, and on the (generalized) spectral density f_w of the continuous time process $w(t)$. We use the notation $m_j(\lambda) = \lambda^{-j}$, and $e_c(\lambda) = e^{i\delta c\lambda}$.

Theorem 1. *We assume that* (9) *holds, and that f_w has d continuous derivatives and is positive and bounded. Then at time $t = \delta\tau + \delta c$, the frf of the optimal filter $\Psi_c(B)$ is given by*

$$\Psi_c(e^{-i\lambda\delta}) = \frac{[g e_c f_w m_{2d}]_\delta(\lambda)}{[f_w m_{2d}]_\delta(\lambda)} \qquad \Psi_c(e^{-i\lambda\delta}) = \frac{i}{1 - e^{-i\lambda\delta}} \frac{[g e_c f_w m_{2d+1}]_\delta(\lambda)}{[f_w m_{2d+2}]_\delta(\lambda)}$$

for the stock (3) *and flow* (4) *cases, respectively.*

For the case when t is a sampled time $t = \delta\tau$, so that $c = 0$, we call the filter $\Psi_0(B)$ the optimal discretization of $\Psi(L)$; also Theorem 1 is then true for stocks under the weaker Assumption (8). At in-between times t with $0 < c < 1$, by setting $\Psi(L)$ equal to the identity filter $\Phi(L) = 1$, Theorem 1 provides the frfs of the filters $\Phi_c(B)$ that yield optimal interpolation. That is, for a Gaussian process (where the linear optimal estimate coincides with the conditional expectation) $E[y(t) \mid Y] = \Phi_c(B)\, y_\tau$. Explicit formulas for $\Phi_c\left(e^{-i\lambda}\right)$ are given below.

Corollary 1. *Under the assumptions of Theorem* 1, *the frf of the filter $\Phi_c(B)$ is given by*

$$\Phi_c(e^{-i\lambda\delta}) = \frac{[e_c f_w m_{2d}]_\delta(\lambda)}{[f_w m_{2d}]_\delta(\lambda)} \qquad \Phi_c(e^{-i\lambda\delta}) = \frac{i}{1 - e^{-i\lambda\delta}} \frac{[e_c f_w m_{2d+1}]_\delta(\lambda)}{[f_w m_{2d+2}]_\delta(\lambda)}$$

for the stock (3) *and flow* (4) *cases, respectively.*

3.3. Optimal Signal Extraction

The next theorem describes the frf of the filter that provides the optimal signal extraction estimate $\Psi_c(B)y_\tau$ of the signal $s(t)$ when the process $y(t)$ has a decomposition (2) with $\mathcal{I}(d)$ signal component $s(t)$ and stationary noise component $n(t)$. It is assumed that $u(t)$ and $n(t)$ are mean zero and uncorrelated with one another—a common assumption in the signal extraction literature. We do not assume that n or u are stochastically continuous—for example, they can be CN. Since $s \in \mathscr{C}^d$ with $s \sim I(d)$, in analogy with (5) we have

$$s(t) = \sum_{j=0}^{d-1} \frac{t^j}{j!} s^{(j)}(0) + [\mathscr{I}^d u](t). \tag{12}$$

Note that (5) holds for $y(t)$ only if $n \in \mathscr{C}^d$, but in general we do not assume this. Were this true, we could write $D^d y(t) = u(t) + D^d n(t)$, called $w(t)$ say, and this would be a well-defined stationary process. If in addition $w \in \mathscr{C}^0$, then its spectral density would be well-defined:

$$f_w(\lambda) = f_u(\lambda) + \lambda^{2d} f_n(\lambda). \tag{13}$$

Now in the general case where $n \notin \mathscr{C}^d$, we still define the non-integrable function f_w via (13), since it still plays a role in determining the signal extraction filter. Of course, f_u and f_n can be well-defined even when $u, n \notin \mathscr{C}^0$, being the Fourier Transforms of the respective autocovariance functions, with transform interpreted in the sense of tempered distributions (Folland, 1995). We only assume that f_u and f_n are bounded. Next, putting (2) and (12) together yields

$$y(t) = \sum_{j=0}^{d-1} \frac{t^j}{j!} s^{(j)}(0) + [I^d u](t) + n(t). \tag{14}$$

Now this form (14) is either stock- or flow-observed, as described by Eqs. (3) or (4). We suppose that when $y(t)$ given by (14) is stock-observed, $n \in \mathscr{C}^0$; in the flow case, we only require that f_n be bounded.

Next, relations (6) and (7) hold in the stock case with s in place of y, and u in place of w; similarly, in the flow case (10) and (11) hold. But these relations do not necessarily hold for y, since we do not assume $y \in \mathscr{C}^d$. Nevertheless, in either the stock or flow case y_τ is reduced to stationarity through d differences, and thus by results in Bell (1984) can be represented in terms of initial values y^* and the well-defined process w_τ. For initial value assumptions, we have

$$y^* \text{ is uncorrelated with } \{u_\tau\}_{\tau=-\infty}^{\infty}, \{n_\tau\}_{\tau=-\infty}^{\infty} \tag{15}$$

$$y^* \text{ is uncorrelated with } \{u(t)\}_{t\in\mathbb{R}}, \{n(t)\}_{t\in\mathbb{R}}. \tag{16}$$

Note that (15) corresponds to the (canonical) Assumption A of Bell (1984); but the stronger assumption (16) is needed to establish optimality when interpolation is being considered, or when we are estimating from flow observations (though see Section 3.4 below). We employ similar notation to that of the previous section, so $t = \delta\tau + \delta c$, etc. The optimal signal extraction filter that accomplishes interpolation is denoted by $\Psi_c(B)$, and its frf is given in the theorem below.

Theorem 2. *We assume that* (16) *holds, and that f_u/f_w has d continuous derivatives and is positive and bounded. Assume that u and n are uncorrelated with one another, mean zero, and weakly stationary, and that $f_u = \mathcal{F}[R_u]$ and $f_n = \mathcal{F}[R_n]$ (with Fourier Transform interpreted in the sense of tempered distributions) are bounded functions. In the stock case, also assume that $n \in \mathcal{C}^0$. Then at time $t = \delta\tau + \delta c$, the frf of the optimal filter $\Psi_c(B)$ is given by*

$$\Psi_c(e^{-i\lambda\delta}) = \frac{[e_c f_u m_{2d}]_\delta(\lambda)}{[f_w m_{2d}]_\delta(\lambda)} \qquad \Psi_c(e^{-i\lambda\delta}) = \frac{i}{1 - e^{-i\lambda\delta}} \frac{[e_c f_u m_{2d+1}]_\delta(\lambda)}{[f_w m_{2d+2}]_\delta(\lambda)}$$

for the stock (3) *and flow* (4) *cases, respectively.*

Remark 1. The formulas in Theorem 2 are obtained from those of Theorem 1 by letting $g = f_u/f_w$, which is the frf of the optimal continuous-lag signal extraction filter described in McElroy and Trimbur (2006). Also, we note the following connection with the results of Bell (1984): in the stock case with $c = 0$, we only need to assume (15)—this is found by examining the proof of Theorem 2—and the given frf is precisely the ratio of pseudo-spectra for signal and data process coming from the stock-discretizations of the continuous-time models. That is, we note that $[f_u m_{2d}]_\delta$ is the spectral density of $\{u_\tau\}$, the differenced stock-sample of the signal process; and $[f_w m_{2d}]_\delta$ is the spectral density of $\{w_\tau\}$, the differenced stock-sample of the data process. Thus Theorem 2 is a natural generalization of classical signal extraction results to handle interpolation, stocks and flows, and generic sampling frequency.

3.4. Computing Filter Coefficients and Flow Estimates

Now in order to obtain the filter coefficients, we must calculate integrals of the frf, i.e., $\psi_k(c) = \frac{\delta}{2\pi}\int_{-\pi/\delta}^{\pi/\delta} \Psi_c(e^{-i\lambda\delta}) e^{ik\lambda\delta}\, d\lambda$. Noting that in general $\frac{\delta}{2\pi}\int_{-\pi/\delta}^{\pi/\delta} [f]_\delta(\lambda) e^{ik\lambda\delta}\, d\lambda = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(\lambda) e^{ik\lambda\delta}\, d\lambda = \mathcal{F}^{-1}[f](\delta k)$, we can compute filter coefficients given $[m_j]_\delta(\lambda)$ for various j. General formulas for these functions are provided in Section A.1 of the Appendix; they are always periodic in λ with period $2\pi/\delta$, and so it follows that for the various cases of stock and flow:

$$\psi_k(c) = \frac{1}{2\pi}\int_{-\infty}^{\infty} g(\lambda) f_w(\lambda) \lambda^{-2d} [f_w m_{2d}]_\delta^{-1}(\lambda) e^{i(k+c)\lambda\delta}\, d\lambda$$

$$\psi_k(c) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{i g(\lambda) f_w(\lambda)}{\lambda^{2d+1}(1 - e^{-i\lambda\delta})} [f_w m_{2d+2}]_\delta^{-1}(\lambda) e^{i(k+c)\lambda\delta}\, d\lambda.$$

(In the case of signal extraction, $g = f_u/f_w$, but the case of a generic filter is also covered by these formulas.) In order to compute the ψ_ks, one must first compute the folded quantities $[f_w m_j]_\delta$, and then use the method of residues to derive the inverse Fourier Transform. This can be quite challenging in certain cases, but in Section 4 we give a few examples where explicit solutions are possible.

The following result gives an alternative formula for the coefficients $\psi_k(c)$, which is more useful when the frf g of the continuous-lag filter is not available, or is difficult to compute from a given kernel ψ. For this result, we suppose that the interpolation filter of Corollary 1 is available, and that $\phi_k(c)$ is known. Although such filter coefficients are defined for $c \in [0, 1)$, we will here extend them for all $c \in \mathbb{R}$ via the following rule:

$$\phi_k(c + j) = \phi_{k+j}(c) \qquad \forall c \in [0, 1), \quad j \text{ integer.} \tag{17}$$

This permits a convenient short-hand for the filters, and is justified as follows: the estimate $\Phi_{c+1}(B)y_\tau$, if defined, should correspond to an interpolation estimate of y at time $t = \delta\tau + \delta(c + 1)$, and thus should be identical to $\Phi_c(B)y_{\tau+1}$. Identifying coefficients yields $\phi_k(c + 1) = \phi_{k+1}(c)$, and by extension, we obtain (17).

Proposition 1. *With the interpolant coefficients extended via* (17), *we have*

$$\psi_k(c) = \delta \int_{-\infty}^{\infty} \psi(\delta z + \delta k)\phi_0(c - z)\, dz.$$

This result holds for either the stock or flow case, by using stock or flow interpolants as appropriate. It is particularly elegant, since it shows that filter discretization occurs as a convolution of the given kernel with the interpolant filter function. As an example, with the "nearest-past neighbor" interpolant $\phi_j(c) = 1_{\{j=0\}}$, we obtain $\psi_k(c) = \int_{\delta c+\delta k-\delta}^{\delta c+\delta k} \psi(z)\, dz$.

Finally, we consider the situation where we are interested in a flow estimate of the form $\int_{\delta\tau-\delta+\delta c}^{\delta\tau+\delta c} \hat{y}(v)\, dv$, rather than the stock form $\hat{y}(\delta\tau + \delta c)$. (The same type of estimate can be obtained for signals $\hat{s}(t)$; in the following discussion, just use $g = f_u/f_w$.) The results of Theorem 1 are easily adapted, and we obtain the filters (for stock and flow data, respectively)

$$\Psi_c(e^{-i\lambda\delta}) = \frac{1 - e^{-i\lambda\delta}}{i} \frac{[g e_c f_w m_{2d+1}]_\delta(\lambda)}{[f_w m_{2d}]_\delta(\lambda)} \qquad \Psi_c(e^{-i\lambda\delta}) = \frac{[g e_c f_w m_{2d+2}]_\delta(\lambda)}{[f_w m_{2d+2}]_\delta(\lambda)}$$

via integration. The results of Corollary 1 and Theorem 2 can be adapted in the same fashion to generating flow estimates. Of particular interest for applications is the flow-signal estimate from flow data, which has frequency response

$$\Psi_c(e^{-i\lambda\delta}) = \frac{[e_c f_u m_{2d+2}]_\delta(\lambda)}{[f_w m_{2d+2}]_\delta(\lambda)}. \tag{18}$$

If one is interested in estimates of this form and $c = 0$, we can relax condition (16) to (15) in Theorem 2, and we see at once that the stated frf exactly matches those of Bell (1984) after flow-discretizing the continuous-time processes for signal and data.

4. ILLUSTRATIONS OF DISCRETIZATION

In this section we consider some applications of Theorem 2 to signal extraction models in econometrics. Specifically, we address the calculation of optimal filters for the Local Level Model (LLM), Smooth Trend Model (STM), and Band-Pass Model (BPM), and also consider a Turning Point (TP) filter for the STM. For reasons of space, we do not include illustrations of Theorem 1, but these can be found in McElroy and Trimbur (2008). Extended derivations of formulas are in the Appendix. When in the frequency domain, we use the notation $z = e^{-i\lambda\delta}$ and $\bar{z} = e^{i\lambda\delta}$.

4.1. LLM

The LLM has the following continuous time formulation:

$$\begin{aligned} Ds(t) &= u(t) \sim WN(q\sigma^2) \\ n(t) &\sim WN(\sigma^2), \end{aligned} \tag{19}$$

where $WN(b)$ denotes CN with spectral density equal to the constant b. Because of the presence of CN $n(t)$ in the underlying process $y(t) = s(t) + n(t)$, it only makes sense to consider flow observations. The flow case of Theorem 2 yields

$$\Psi_c(e^{-i\lambda\delta}) = \frac{3\delta\left((e^{i\lambda\delta}+1) - c^2|1-e^{-i\lambda\delta}|^2 e^{i\lambda\delta} - 2c(1-e^{i\lambda\delta})\right)}{\delta^2(2\cos\lambda\delta + 4) + 6|1-e^{-i\lambda\delta}|^2/q}.$$

If we are interested in a flow-signal, then (18) yields

$$\Psi_c(e^{-i\lambda\delta}) = \frac{\begin{aligned}\delta^2\big((2\cos\lambda\delta + 4) + c^3(1-e^{-i\lambda\delta})(1-e^{i\lambda\delta})^2 \\ - 3c^2|1-e^{-i\lambda\delta}|^2 + 3c(e^{i\lambda\delta} - e^{-i\lambda\delta})\big)\end{aligned}}{\delta^2(2\cos\lambda\delta + 4) + 6|1-e^{-i\lambda\delta}|^2/q}.$$

Letting $\eta_1 \in (0,1)$ be defined in terms of δ and q and given in Appendix A.3, the coefficients in the case of a flow-signal are

$$\psi_j = \begin{cases} \eta_1^{-j-2} r_-^f & j \leq -2 \\ \dfrac{\delta^2 c^3}{\delta^2 - 6/q} + \eta_1^{-1} r_-^f & j = -1 \\ \dfrac{\delta^2 (1-c)^3}{\delta^2 - 6/q} + \eta_1^{-1} r_+^f & j = 0 \\ \eta_1^{j-1} r_+^f & j \geq 1, \end{cases}$$

where the constants r_-^f and r_+^f are given in Appendix A.3. In the case of a stock-signal (as in Theorem 2) the coefficients are

$$\psi_j = \begin{cases} \eta_1^{-j-2} r_-^s & j \leq -2 \\ \dfrac{3\delta c^2}{\delta^2 - 6/q} + \eta_1^{-1} r_-^s & j = -1 \\ \eta_1^{j} r_+^s & j \geq 0, \end{cases}$$

with stock constants r_-^s and r_+^s given in Appendix A.3.

4.2. STM

The smooth trend model has the following continuous time formulation:

$$D^2 s(t) = u(t) \sim WN(q\sigma^2)$$
$$n(t) \sim WN(\sigma^2).$$

As before, we must consider only flow sampling; the flow case of Theorem 2 yields

$$\Psi_c(e^{-i\lambda\delta}) = 5\delta^3 \left[\begin{array}{l} c^4 \bar{z}|1-z|^4 + 4c^3(1-\bar{z})|1-z|^2 + 6c^2(1-\bar{z})(z-\bar{z}) \\ \quad - 8c(1+z)(1-\bar{z}^2) + 4c(z+\bar{z})(1-\bar{z}) + 12\bar{z}(1+z) \end{array} \right]$$
$$\cdot \left(\delta^4(66 + 26z + 26\bar{z} + z^2 + \bar{z}^2) + 120|1-z|^4/q\right)^{-1}.$$

For the case of a flow-signal (18), we have

$$\Psi_c(e^{-i\lambda\delta}) = \delta^4 \left[\begin{array}{l} c^5 \frac{|1-z|^6}{z-1} + 5c^4|1-z|^4 + 10c^3(z-\bar{z})|1-z|^2 + 20c^2(z-\bar{z})^2 \\ \quad + 10c^2(z+\bar{z})|1-z|^2 - 60c(z-\bar{z}) \\ \quad + 5c(z-\bar{z})|1-z|^2 + \left(66 + 26z + 26\bar{z} + z^2 + \bar{z}^2\right) \end{array} \right]$$
$$\cdot \left(\delta^4(66 + 26z + 26\bar{z} + z^2 + \bar{z}^2) + 120|1-z|^4/q\right)^{-1}.$$

The coefficient formulas are quite complicated. For $v_2 \in (0,1)$ and v_3, v_4 complex conjugate with unit modulus given in Appendix A.3, the coefficients in the flow-signal case are given by

$$\begin{aligned}
\psi_j &= \frac{\Theta_c(v_2) v_2^{-j-2}}{(\delta^4 - 120/q)(v_2 - v_1)(v_2^2 + \beta v_2 + 1)} \\
&\quad + \frac{\Theta_c(v_3) v_3^{-j-2}}{2(\delta^4 - 120/q)(v_3 - v_4)(v_3^2 + \alpha v_3 + 1)} \\
&\quad + \frac{\Theta_c(v_4) v_4^{-j-2}}{2(\delta^4 - 120/q)(v_4 - v_3)(v_4^2 + \alpha v_4 + 1)} \qquad j \leq -2, \\
\psi_{-1} &= \frac{\Theta_c(0)}{\delta^4 - 120/q} + \frac{\Theta_c(v_2) v_2^{-1}}{(\delta^4 - 120/q)(v_2 - v_1)(v_2^2 + \beta v_2 + 1)} \\
&\quad + \frac{\Theta_c(v_3) v_3^{-1}}{2(\delta^4 - 120/q)(v_3 - v_4)(v_3^2 + \alpha v_3 + 1)} \\
&\quad + \frac{\Theta_c(v_4) v_4^{-1}}{2(\delta^4 - 120/q)(v_4 - v_3)(v_4^2 + \alpha v_4 + 1)}, \\
\psi_0 &= \frac{\Phi_c(0)}{\delta^4 - 120/q} + \frac{\Phi_c(v_2) v_2^{-1}}{(\delta^4 - 120/q)(v_2 - v_1)(v_2^2 + \beta v_2 + 1)} \\
&\quad + \frac{\Phi_c(v_3) v_3^{-1}}{2(\delta^4 - 120/q)(v_3 - v_4)(v_3^2 + \alpha v_3 + 1)} \\
&\quad + \frac{\Phi_c(v_4) v_4^{-1}}{2(\delta^4 - 120/q)(v_4 - v_3)(v_4^2 + \alpha v_4 + 1)}, \\
\psi_j &= \frac{\Phi_c(v_2) v_2^{j-1}}{(\delta^4 - 120/q)(v_2 - v_1)(v_2^2 + \beta v_2 + 1)} \\
&\quad + \frac{\Phi_c(v_3) v_3^{j-1}}{2(\delta^4 - 120/q)(v_3 - v_4)(v_3^2 + \alpha v_3 + 1)} \\
&\quad + \frac{\Phi_c(v_4) v_4^{j-1}}{2(\delta^4 - 120/q)(v_4 - v_3)(v_4^2 + \alpha v_4 + 1)} \qquad j \geq 1.
\end{aligned}$$

These depend on polynomials Θ_c and Φ_c, as well as constants α and β, that are given in Appendix A.3. Now for stock-signal case the roots v_j happen to be the same (but Θ_c and Φ_c are different), and the coefficients are given by

$$\psi_j = \frac{\Theta_c(v_2) v_2^{-j-2}}{(\delta^4 - 120/q)(v_2 - v_1)(v_2^2 + \beta v_2 + 1)}$$

$$+\frac{\Theta_c(v_3)v_3^{-j-2}}{2(\delta^4-120/q)(v_3-v_4)(v_3^2+\alpha v_3+1)}$$

$$+\frac{\Theta_c(v_4)v_4^{-j-2}}{2(\delta^4-120/q)(v_4-v_3)(v_4^2+\alpha v_4+1)} \qquad j\leq -2,$$

$$\psi_{-1}=\frac{5c^4\delta^3}{\delta^4-120/q}+\frac{\Theta_c(v_2)v_2^{-1}}{(\delta^4-120/q)(v_2-v_1)(v_2^2+\beta v_2+1)}$$

$$+\frac{\Theta_c(v_3)v_3^{-1}}{2(\delta^4-120/q)(v_3-v_4)(v_3^2+\alpha v_3+1)}$$

$$+\frac{\Theta_c(v_4)v_4^{-1}}{2(\delta^4-120/q)(v_4-v_3)(v_4^2+\alpha v_4+1)},$$

$$\psi_j=\frac{\Phi_c(v_2)v_2^{j-1}}{(\delta^4-120/q)(v_2-v_1)(v_2^2+\beta v_2+1)}$$

$$+\frac{\Phi_c(v_3)v_3^{j-1}}{2(\delta^4-120/q)(v_3-v_4)(v_3^2+\alpha v_3+1)}$$

$$+\frac{\Phi_c(v_4)v_4^{j-1}}{2(\delta^4-120/q)(v_4-v_3)(v_4^2+\alpha v_4+1)} \qquad j\geq 0.$$

All of these formulas are derived in the Appendix.

4.3. TP Filters

If a given continuous time filter $\Psi(L)$ produces a smooth trend estimate, then the filter with frequency response $i\lambda\Psi(e^{-i\lambda})$ estimates the velocity of that trend (see McElroy and Trimbur, 2006) for a discussion). When the velocity changes sign, this indicates a turning point in the estimated trend. Here we apply this concept to the flow case of the STM. Using Theorem 2, we see that the frf of the discretized TP filter is

$$\Psi_c(e^{-i\lambda\delta})=\frac{i[e_c m_5]_\delta(\lambda)}{[m_6]_\delta(\lambda)+[m_2]_\delta(\lambda)/q},$$

which is just $1-e^{-i\lambda\delta}$ times the frf for the stock-signal case of the STM. This results in the following coefficients (where the constants have the same definition as in the discussion of the STM):

$$\psi_j=\frac{\Theta_c(v_2)(1-v_2)v_2^{-j-2}}{(\delta^4-120/q)(v_2-v_1)(v_2^2+\beta v_2+1)}$$

$$+ \frac{\Theta_c(v_3)(1-v_3)v_3^{-j-2}}{2(\delta^4 - 120/q)(v_3 - v_4)(v_3^2 + \alpha v_3 + 1)}$$

$$+ \frac{\Theta_c(v_4)(1-v_4)v_4^{-j-2}}{2(\delta^4 - 120/q)(v_4 - v_3)(v_4^2 + \alpha v_4 + 1)} \qquad j \leq -2,$$

$$\psi_{-1} = \frac{5c^4\delta^3}{\delta^4 - 120/q} + \frac{\Theta_c(v_2)(1-v_2)v_2^{-1}}{(\delta^4 - 120/q)(v_2 - v_1)(v_2^2 + \beta v_2 + 1)}$$

$$+ \frac{\Theta_c(v_3)(1-v_3)v_3^{-1}}{2(\delta^4 - 120/q)(v_3 - v_4)(v_3^2 + \alpha v_3 + 1)}$$

$$+ \frac{\Theta_c(v_4)(1-v_4)v_4^{-1}}{2(\delta^4 - 120/q)(v_4 - v_3)(v_4^2 + \alpha v_4 + 1)},$$

$$\psi_0 = \frac{10\delta^3 c(1-c^2)(3c+2)}{\delta^4 - 120/q} + \frac{\Theta_c(v_2)(1-v_2)v_2^{-2}}{(\delta^4 - 120/q)(v_2 - v_1)(v_2^2 + \beta v_2 + 1)}$$

$$+ \frac{\Theta_c(v_3)(1-v_3)v_3^{-2}}{2(\delta^4 - 120/q)(v_3 - v_4)(v_3^2 + \alpha v_3 + 1)}$$

$$+ \frac{\Theta_c(v_4)(1-v_4)v_4^{-2}}{2(\delta^4 - 120/q)(v_4 - v_3)(v_4^2 + \alpha v_4 + 1)},$$

$$\psi_j = \frac{\Phi_c(v_2)(v_2 - 1)v_2^{j-2}}{(\delta^4 - 120/q)(v_2 - v_1)(v_2^2 + \beta v_2 + 1)}$$

$$+ \frac{\Phi_c(v_3)(v_3 - 1)v_3^{j-2}}{2(\delta^4 - 120/q)(v_3 - v_4)(v_3^2 + \alpha v_3 + 1)}$$

$$+ \frac{\Phi_c(v_4)(v_4 - 1)v_4^{j-2}}{2(\delta^4 - 120/q)(v_4 - v_3)(v_4^2 + \alpha v_4 + 1)} \qquad j \geq 1.$$

4.4. BP and LPM Filters

The trend-cycle-irregular model in continuous-time is discussed in Harvey and Stock (1993) and Chambers and McGarry (2002). In McElroy and Trimbur (2006), continuous-lag BP and LP filters are derived. The underlying model has a trend component m, cycle component c, and irregular component i, which have the following CARIMA models:

$$Dm(t) = u(t) \sim WN(q\sigma^2)$$

$$i(t) \sim WN(\sigma^2)$$

$$(D + \rho + i\lambda_c)(D + \rho - i\lambda_c)c(t) \sim WN(r\sigma^2).$$

We must consider only flow sampling because of the continuous white noise irregular. Here q and r are positive signal-noise ratios, and ρ indicates the strength of the cycle peak in f_c. Values of ρ closer to zero indicate a higher peak, with the location of the maximum occurring close to $\pm\lambda_c$. The component pesudo-spectra are

$$f_m(\lambda) = \frac{q\sigma^2}{\lambda^2} \qquad f_i(\lambda) = \sigma^2 \qquad f_c(\lambda) = \frac{r\sigma^2}{(\rho^2 + (\lambda - \lambda_c)^2)(\rho^2 + (\lambda + \lambda_c)^2)}.$$

Using the calculus of residues, we find that

$$R_c(x) = \frac{r\sigma^2 e^{-\rho|x|}}{4\rho\lambda_c(\lambda_c^2 + \rho^2)} Re\left(e^{i\lambda_c|x|}(\lambda_c - \rho i)\right).$$

The BP filter that we consider is the signal extraction filter for a process with a cyclical signal and a trend-irregular noise. So we apply Theorem 2 with the role of signal and noise swapped (so that the signal is stationary and the noise is nonstationary), which is only a change in nomenclature. The LP filter corresponds to a trend signal with cycle-irregular noise. Focusing on the case of a flow-signal (18), the BP and LP frequency response functions, respectively, are given by

$$\Psi_c(e^{-i\lambda\delta}) = \frac{[e_c f_c m_4]_\delta(\lambda)}{[qm_4 + m_2 + f_c m_4]_\delta(\lambda)} \qquad \Psi_c(e^{-i\lambda\delta}) = \frac{[e_c q m_4]_\delta(\lambda)}{[qm_4 + m_2 + f_c m_4]_\delta(\lambda)}.$$

The explicit expression is complicated. Formulas in Appendix A.1 can be used to compute $[e_c m_4]$ and so forth; $[e_c f_c m_4]_\delta$ is given below:

$$\frac{r\sigma^2}{6(\rho^2+\lambda_c^2)^2}\left(\begin{array}{l}\dfrac{12\delta(\lambda_c^2-\rho^2)}{(\rho^2+\lambda_c^2)^2}\left(\dfrac{c}{z-1}+\dfrac{1}{|1-z|^2}\right)\\ -\delta^3\left(\dfrac{c^3}{z-1}+\dfrac{3c^2}{|1-z|^2}+\dfrac{3c(z-\bar z)}{|1-z|^4}+\dfrac{2(z-\bar z)^2}{|1-z|^6}+\dfrac{z+\bar z}{|1-z|^4}\right)\end{array}\right)$$
$$-\frac{r\sigma^2}{8\lambda_c\rho}\left(\begin{array}{l}(\lambda_c+i\rho)^{-5}\dfrac{\begin{array}{c}z(e^{i\delta\lambda_c(c-1)}e^{-\delta\rho(c-1)}-e^{-i\delta\lambda_c(c-1)}e^{\delta\rho(c-1)})\\ -(e^{i\delta\lambda_c c}e^{-\delta\rho c}-e^{-i\delta\lambda_c c}e^{\delta\rho c})\end{array}}{(z-e^{\delta(\rho-i\lambda_c)})(z-e^{-\delta(\rho-i\lambda_c)})}\\ -(\lambda_c-i\rho)^{-5}\dfrac{\begin{array}{c}z(e^{i\delta\lambda_c(c-1)}e^{\delta\rho(c-1)}-e^{-i\delta\lambda_c(c-1)}e^{-\delta\rho(c-1)})\\ -(e^{i\delta\lambda_c c}e^{\delta\rho c}-e^{-i\delta\lambda_c c}e^{-\delta\rho c})\end{array}}{(z-e^{-\delta(\rho+i\lambda_c)})(z-e^{\delta(\rho+i\lambda_c)})}\end{array}\right).$$

This is derived in Appendix A.3. Now for a stock-signal, we have

$$\Psi_c(e^{-i\lambda\delta}) = \frac{i[e_c f_c m_3]_\delta(\lambda)}{(1-z)[qm_4 + m_2 + f_c m_4]_\delta(\lambda)}$$
$$\Psi_c(e^{-i\lambda\delta}) = \frac{i[e_c q m_3]_\delta(\lambda)}{(1-z)[qm_4 + m_2 + f_c m_4]_\delta(\lambda)}.$$

The formula for $[e_c f_c m_3]_\delta$ is given below:

$$\frac{i\delta^2 r\sigma^2}{2(\rho^2+\lambda_c^2)^2}\left[\frac{c^2}{z-1} + \frac{2c}{|1-z|^2} + \frac{(z-\bar{z})}{|1-z|^4}\right] - \frac{2ir\sigma^2(\lambda_c^2-\rho^2)}{(\rho^2+\lambda_c^2)^4(z-1)}$$
$$-\frac{r\sigma^2}{8\lambda_c\rho}\left(\begin{array}{c} (\lambda_c+i\rho)^{-4}\dfrac{\begin{array}{c} z(e^{i\delta\lambda_c(c-1)}e^{-\delta\rho(c-1)} + e^{-i\delta\lambda_c(c-1)}e^{\delta\rho(c-1)}) \\ -(e^{i\delta\lambda_c c}e^{-\delta\rho c} + e^{-i\delta\lambda_c c}e^{\delta\rho c}) \end{array}}{(z-e^{\delta(\rho-i\lambda_c)})(z-e^{-\delta(\rho-i\lambda_c)})} \\ -(\lambda_c-i\rho)^{-4}\dfrac{\begin{array}{c} z(e^{i\delta\lambda_c(c-1)}e^{\delta\rho(c-1)} + e^{-i\delta\lambda_c(c-1)}e^{-\delta\rho(c-1)}) \\ -(e^{i\delta\lambda_c c}e^{\delta\rho c} + e^{-i\delta\lambda_c c}e^{-\delta\rho c}) \end{array}}{(z-e^{-\delta(\rho+i\lambda_c)})(z-e^{\delta(\rho+i\lambda_c)})} \end{array} \right).$$

So the frfs for the BP and LP filters are fairly complicated, and at present the coefficients must be determined by numerical integration of these functions.

5. EMPIRICAL APPLICATION

Here we apply a trend estimator to the inflation rate. The consumer price index (Source: Bureau of Labor Statistics) is widely reported in the media and closely watched by policymakers and analysts. Sometimes, however, the index changes rapidly and undergoes sudden transitions, so especially at such times it is useful to measure the underlying signal in the movements. Thus, we can use the trend estimator to filter out the uninformative noise, and by estimating the model, we can extract the signal in a way that adapts to the properties of the series.

The prices of some goods, such as commodities, may change continuously, while other products may have a menu updated periodically. Clearly, the price level representative of a large economy evolves constantly. Consider a simple model for the instantaneous rate of change: $d\log P(t)/dt = s(t) + n(t)$, where signal and noise have the form (19). Given a sampling interval δ, the inflation rate at time $t = \delta\tau$ is $r_{\delta\tau} = (100/\delta) \times \log(1 + r^*_{\delta\tau})$, where $r^*_{\delta\tau}$ is the straight percent change in prices from $t-\delta$ to t. That is, $r^*_{\delta\tau} = (P_t - P_{t-\delta})/P_{t-\delta} - 1$. This definition follows Stock and Watson (1999). Using this transformation, analogous to taking log-returns for asset prices, allows for continuous-time analysis independent of sampling frequency. Now the inflation rate may be written as (using $t = \delta\tau$)

$$
\begin{aligned}
r_{\delta\tau} &= (100/\delta) \times \log[P(t)/P(t-\delta)] = (100/\delta) \times (\log P(t) - \log P(t-\delta)) \\
&= (100/\delta) \int_{t-\delta}^{t} d \log P(t)/dt,
\end{aligned}
$$

so the observations $r_{\delta\tau}$ are constructed as in (4). That is, the flow observations are based on cumulative sampling of the continuous process $d \log P_t/dt$. To estimate the continuous-time signal-noise ratio q, we use a sample at the highest available frequency, that is, monthly. In this case, $r_{\delta\tau}$ is the δ-interval inflation rate for $\delta = 1/12$. The sample period is January 1960 to January 2009; monthly CPI inflation is shown in Fig. 1. To construct the likelihood function, we set up the discrete unobserved components model. It is shown in Harvey and Trimbur (2007) that, with the timing shifts, $\mu_\tau^* = \mu(t_{\tau-1})$ and $\eta_\tau^* = \eta_{\tau-1}$; also, let $r_{\delta\tau} = s_\tau^* + n_\tau^*$ and $s_{\tau+1}^* = s_\tau^* + u_\tau^*$ with covariance matrix

$$
Var\begin{bmatrix} u_\tau^* \\ n_\tau^* \end{bmatrix} = \begin{bmatrix} \delta\sigma_u^2 & \frac{1}{2}\delta\sigma_u^2 \\ \frac{1}{2}\delta\sigma_u^2 & \frac{1}{3}\delta\sigma_u^2 + \sigma^2/\delta \end{bmatrix}.
$$

This has the usual form of the discrete-time local level model but with (positively) correlated disturbances. We used a program written in the Ox language (Doornik, 2007) with the SsfPack routines (Koopman et al., 1999) that estimated the model by applying maximum likelihood to the state space. The estimates of the continuous-time variance parameters are $\sigma_u^2 = 6.138$, $\sigma_n^2 = 0.5385$. Taking the ratio $q = \sigma_u^2/\sigma_n^2$ gives a signal-noise ratio of $q = 11.40$. The discrete-time signal-noise ratio is $q_\delta^* = \frac{Var(u_\tau^*)}{Var(n_\tau^*)} = \frac{\delta^2 q}{\delta^2 q/3+1}$, though its role is slightly changed compared to an uncorrelated components model. The estimate $q_\delta^* = 0.0812$ is close to the value for an estimated uncorrelated local level model in discrete-time, $q_\delta^\dagger = 0.0802$, but the slight correlation has some effect, so that we can adapt the filter precisely based on our estimate of the underlying q. For discrete-time models, the dependence of trend filters on signal-noise ratios was illustrated in Harvey and Koopman (2000). McElroy and Trimbur (2006) show the same relationships for continuous-lag filters based on the q parameter. The interpretation is that lower q indicates more noise, so more smoothing is needed, with the coefficients decaying less rapidly. On the other hand, a higher q indicates more variability in the signal, so more information is attached to its nearby movements, with the coefficients declining fast away from the center weight.

The signal refers to the instantaneous inflation rate with the noise removed. The filters are theoretically designed to apply to doubly infinite series, so we will approximate them via truncation to ten years of observations on either side. Toward the ends of the sample, however, we do not have access to all the observations needed as inputs to the filters. Following Findley et al. (1998), this endpoint problem is solved by forecasting and backcasting the series near the beginning and end. We determine the forecasting model by exploration with different STS models. In particular, the model has the form:

$$y_t = \mu_t + \psi_{2,t} + v_t + \varepsilon_t, \qquad \varepsilon_t \sim WN(0, \sigma_\varepsilon^2), \quad t = 1, \ldots, T \qquad (20)$$

where μ_t is the random walk trend, $\psi_{2,t}$ a second order cycle (see Harvey et al., 2007) and v_t an AR(2) component. We chose this model among several alternatives on the basis of fit and diagnostics. The reason for including the second order, instead of the first order, cycle is to generate a plausible procyclical component; the estimated period is about 5.4 years. The AR(2) component captures the additional serial correlation and leads to a much lower Box–Ljung Q-statistic. The interpretation behind the autoregressive component is based on positive momentum, where unusually high inflation in one period is often followed by high inflation the next. The R_d^2, or coefficient of determination with respect to first differences (this measures the within sample explanatory power of the model beyond that of a random walk with drift), is about 27%, a favorable result for a univariate model. See Fig. 1 for the recent behavior together

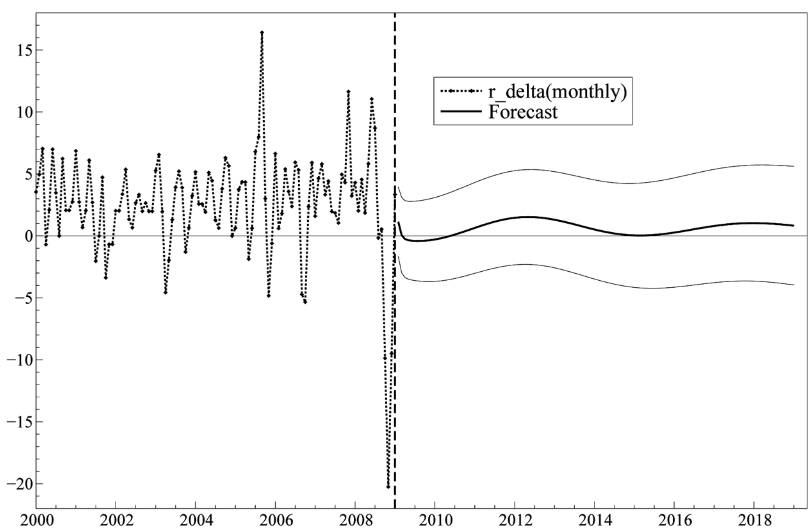

FIGURE 1 Forecast extension for monthly CPI inflation. The sample period is January 1960 to January 2009, and the forecast horizon goes out to ten years. Only months January 2000 through January 2019 are displayed.

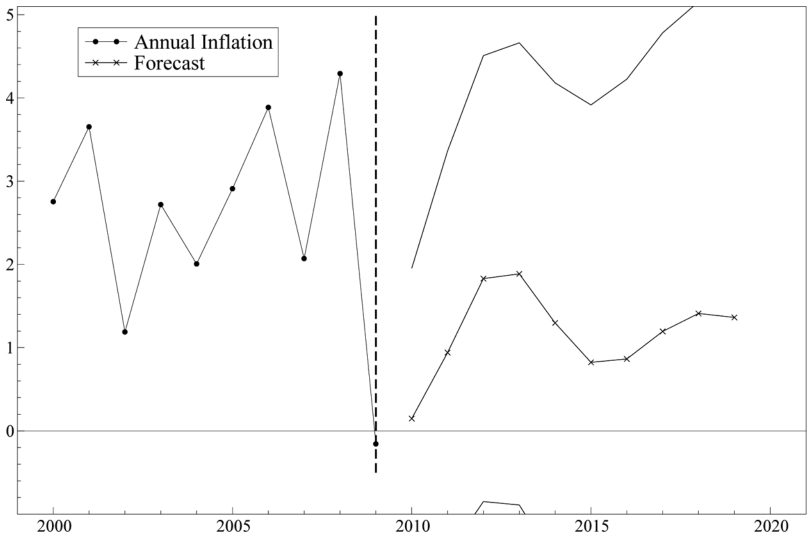

FIGURE 2 Forecast extension for annual CPI inflation. The sample period is 1960 to 2009, and the forecast horizon goes out ten years. Only years 2000 through 2019 are displayed.

with the forecast, which calls for a temporary slip in monthly inflation followed by a slow, cyclical rise moving into 2012.

The media also commonly reports the annual rate of inflation, or the change from the level one year ago; in our framework this means $\delta = 1$. These estimates are considerably smoother than the monthly rates. The most recent observation of the series, Jan. 09 (log-multiple change from Jan. 08), slips just below zero, while the forecast extension has slow oscillations above zero with the same period as for monthly data. See Fig. 2. We generated this forecast from an STS model for annual inflation; the model has the same form as (20) except that it omits the AR(2), which appear unnecessary in the annual. Again the diagnostics are favorable with a low Q-statistic and a R_d^2 of about 18%. With an estimated period of about five and a half years, the monthly and annual models seem to capture the same underlying cyclical factors.

To coherently filter the monthly and annual series, we apply the flow observation-stock estimate case of Theorem 2 to both the monthly and annual flow series. The trend filters were applied for both $\delta = 1, 1/12$ (with edge effects handled by forecasts and backcasts as noted above). Since the target signal is a stock, the annual trend should be close to every twelfth value of the monthly trend. We also produced interpolated annual trend values at increments of $1/12$, i.e., letting $c = 0, 1/12, 2/12, \ldots, 11/12$ corresponding to monthly interpolations. These interpolated annual trend values should be a close match to the monthly trend values, and both trends are superimposed in Fig. 3.

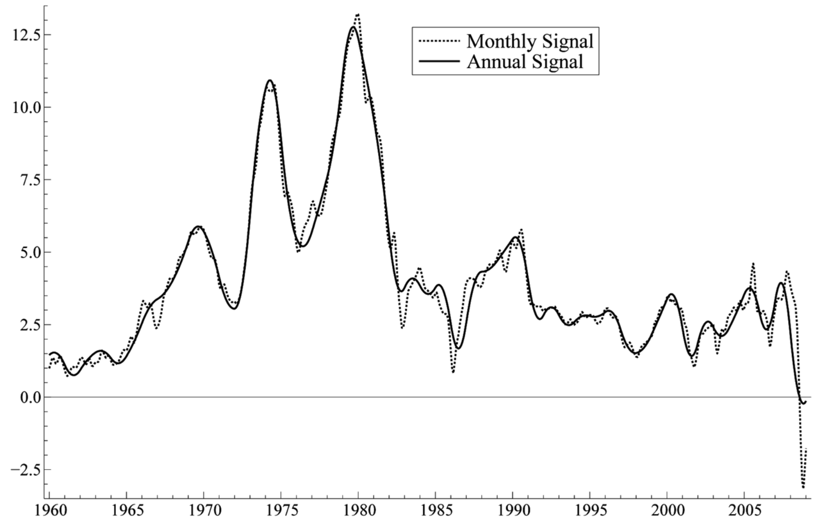

FIGURE 3 Estimated signal for monthly and annual CPI inflation data.

The figure shows that, away from the beginning and end of the sample period, the monthly and annual trend estimates nearly match, yet some disagreement remains (it is clear from the proof of Theorem 2 that agreement is exact only when the stochastic component of the data is absent, i.e., the data is purely deterministic and linear). The discrepancy between the estimates also arises from the monthly fluctuations that are absent from the annual data; for instance, around the mid-1980s the monthly data showed short-term volatility without a sustained movement in trend. The monthly signal responds more to these higher frequency movements. Though the annual signal involves the use of less information, it has the advantage of picking out the longer-term pattern. This tendency becomes apparent at the end of the sample, where the annual signal suggests a very moderate deflation, while the monthly signal has responded to the large setback due to the collapse in the oil price. Overall, the agreement between the two trends is marvelously close, powerfully illustrating the coherency aspect of this article (statistically, there was no asymmetry or systematic time-delay between the two trends). The greater smoothness of the annual interpolated trend is due to the quadratic nature of the interpolations (see Section 4.1), where the degree of the interpolating polynomial is in general determined by d and the pseudo-spectrum of the signal (see Theorem 2).

With a continuous trend filter we can get estimates at all times, so we could address questions such as when turning points have occurred. For example, one might have an interest in how the rise in inflation during the mid- to late-1980s transpired. The interpolation of the annual signal indicates that core inflation reached a trough of 1.68% in July 1986 and gained steadily until it reached a peak of 5.52% in March 2000. The weights evolve as the interpolant changes between 0 and 1 according to the formulas derived earlier. We could also plot weekly underlying inflation by taking c in the interpolation as multiples of 1/52, for both annual and monthly trend filters.

For brevity we omit further extensions here. In some experiments, we found that monthly inflation has a small seasonal component, which could be modeled explicitly or removed by seasonal adjustment with the X-12-ARIMA program. Proceeding in this direction, one could imagine filter discretization for seasonal adjustment or for processes with seasonal or more general unit roots.

Here, we have considered the local level model as a simple benchmark for inflation. The local level model can be fit to the series, so this forms a basis for adaptive filter design. The forecasting and backcasting accounted for the additional serial correlation in the inflation rate by including some kind of cyclical component. The inflation rate also appears non-Gaussian and heteroskedastic, so heavy-tailed disturbances and stochastic volatility are other possible extensions. In any case, in estimating a reasonable signal near the end-point, it seems important to account for the cyclical pattern so as to give plausible forecasts as input to the filter.

In filter design, the local level model demonstrates the flexibility of our approach through the weights' explicit dependence on δ, on the underlying parameters, and on the interpolant c. Filters designed from more sophisticated models would be based on the same principles. Another extension would be to use an auxiliary indicator, such as commodity price data, available at daily or weekly intervals to refine the interpolation.

6. CONCLUSIONS

This article has developed a new method for filter design for economic time series. In particular, to handle the different kinds of data that can arise, which are observed as stock or flow and which may have different sampling frequencies, we suggest the use of an underlying filter set up in continuous-time, or continuous-lag filter. New mathematical results in Section 3 handle filtering, signal extraction, and interpolation in an MSE optimal fashion. Explicit formulas for certain popular models have been given in Section 4, including the LLM and continuous-time HP filter.

Using these results, it is possible to produce trend estimates for stock and flow time series measured at several different sampling frequencies, such that the output series are in close agreement at interpolated times; this is illustrated on CPI in Section 5. This type of technology brings about an automatic benchmarking in situations where it is necessary for higher frequency trends to "match up" to trends computed from more highly aggregated data.

This problem of filter design and the related topic of interpolation, or estimation between observation points, has been addressed in earlier work by first deriving the discrete model and second using discrete-time theory. We view our results as complementary to work such as Harvey and Stock (1993), since knowledge of the discrete-time formulation may prove useful beyond the filtering problem. We suggest, however, a more direct route to filter design that preserves consistency of the signal estimation. Our results also open the door to new applications, such as velocity filters for turning point analysis. Another possible area of investigation is seasonality where the coherent filters would handle monthly and quarterly, and perhaps weekly, data in a consistent framework.

APPENDIX

The material in this Appendix is organized in correspondence with the order of topics in the main text. Hence we first discuss folds, then we give the proofs of Theorems 1 and 2, and then we give details on frf and coefficient calculations for the applications in Section 4.

A.1. Computing Folds

Much is known about folds in the engineering literature: see Solo (1983), and the literature on Laplace/Z Transform pairs in control engineering (Jury, 1973; Kuo, 1963). The innovation here is that we incorporate the sampling frequency δ and the interpolant c into the discussion, and make distinctions between stock and flow sampling. Suppose that a mean zero stationary continuous time process $x(t)$ has orthogonal increments representation (Brockwell and Davis, 1991) $x(t) = \int_{-\infty}^{\infty} e^{it\lambda}\, d\mathbb{Z}(\lambda)$ where $\mathbb{E}[d\mathbb{Z}(\lambda)\overline{d\mathbb{Z}}(\lambda)] = f(\lambda)\, d\lambda/2\pi$. For example, if x is stochastically continuous, then Theorem 4.11.1 of Priestley (1981) guarantees the existence of such a representation. If we stock observe x via the equation $x_\tau = x(\delta\tau)$, where τ is integer, then

$$x_\tau = \sum_{l=-\infty}^{\infty} \int_{(2l-1)\pi/\delta}^{(2l+1)\pi/\delta} e^{i\tau\lambda\delta}\, d\mathbb{Z}(\lambda) = \int_{-\pi/\delta}^{\pi/\delta} e^{i\tau\lambda\delta} \sum_{l=-\infty}^{\infty} d\mathbb{Z}(\lambda + 2\pi l/\delta)$$

by change of variable. Likewise, the autocovariances are only considered at lags δh where h is now integer, and we can write $R_h = R(\delta h) = \frac{1}{2\pi} \int_{-\pi/\delta}^{\pi/\delta} e^{ih\lambda\delta} \sum_{l=-\infty}^{\infty} f(\lambda + 2\pi l/\delta)\, d\lambda$. Hence the spectral density of x_τ is given by $\sum_{h=-\infty}^{\infty} R_h e^{-ih\lambda\delta} = \frac{1}{\delta} \sum_{l=-\infty}^{\infty} f(\lambda + 2\pi l/\delta)$ for $\lambda \in [-\pi/\delta, \pi/\delta]$; this latter formula is denoted by the notation $[f]_\delta(\lambda)$. It is called the "fold" of f, since it is obtained—graphically speaking—by chopping f up into contiguous domains of size $[-\pi/\delta, \pi/\delta]$ and overlaying the corresponding functions (see Koopmans, 1974, for a picture). It is obvious from the above equation that the fold is periodic if we view $[f]_\delta$ as a function on the real line. In order to be well defined, it is necessary that the tails of f decay faster than $1/\lambda$; methods for computing folds from a given f are discussed below. Now from the expression for $[f]_\delta(\lambda)$, we see that the highest frequency that can be observed is π/δ, and at any λ the value of the spectrum $[f]_\delta(\lambda)$ is confounded by the aliases $f(\lambda + 2\pi l/\delta)$ for all integers l. The frequency $2\pi/\delta$ is referred to as the Nyquist folding frequency (Blackman and Tukey, 1958).

For a flow observation of $x(t)$, we obtain $x_\tau = \int_{-\infty}^{\infty} e^{i\delta\tau\lambda}(1 - e^{-i\lambda\delta})(i\lambda)^{-1} d\mathbb{Z}(\lambda)$, with autocovariance function $R_h = R(\delta h) = \frac{1}{2\pi}\int_{-\pi/\delta}^{\pi/\delta} e^{ih\lambda\delta}|1 - e^{-i\lambda\delta}|^2 \sum_{l=-\infty}^{\infty} \frac{f(\lambda+2\pi l/\delta)}{(\lambda+2\pi l/\delta)^2}\, d\lambda$. The corresponding spectral density is $[fm_2]_\delta(\lambda)|1 - e^{-i\lambda\delta}|^2$. In several places in Section 4, we need to calculate $[e_c m_j]_\delta(\lambda)$ for various $j \geq 2$. When $c = 0$ and $\lambda \neq 0$, we can use Theorem 4.9a of Henrici (1974) as follows:

$$[m_j]_\delta(\lambda) = \frac{1}{\delta}\sum_{h=-\infty}^{\infty} (\lambda + 2\pi h/\delta)^{-j} = -\frac{1}{\delta}\sum_{\zeta} Res((\lambda + 2\pi \cdot /\delta)^{-j} a, \zeta),$$

where the sum is over all the poles ζ of the function $(\lambda + 2\pi \cdot /\delta)^{-j}$, and *Res* denotes the residue. The function a is defined by $a(x) = \pi \cot \pi x$. This formula does not cover the case that $\lambda = 0$, since the fold explodes to infinity at that frequency. The above application of Theorem 4.9a of Henrici (1974) can be made to compute the fold of any function g, so long as it is a rational function of λ with a zero of order at least two at infinity. In our case, we have a pole of order j at $-\delta\lambda/2\pi$, so the fold is given by $[m_j]_\delta(\lambda) = -\frac{1}{\delta(j-1)!}\left(\frac{\delta}{2\pi}\right)^j a^{(j-1)}\left(\frac{-\delta\lambda}{2\pi}\right)$. We pursue a method of calculation for this below, which is easier than a brute force approach. In the case that $c \neq 0$, we instead apply Theorem 4.9b of Henrici (1974) (which also covers $c = 0$, but typically involves more computations):

$$\begin{aligned}[e_c m_j]_\delta(\lambda) &= \frac{1}{\delta}\sum_{h=-\infty}^{\infty} (\lambda + 2\pi h/\delta)^{-j} e^{i\delta c(\lambda + 2\pi h/\delta)} \\ &= -\frac{e^{i\delta\lambda c}}{\delta}\sum_{\zeta} Res((\lambda + 2\pi \cdot /\delta)^{-j}\chi, \zeta),\end{aligned}$$

where the sum is over all the poles ζ of the function $(\lambda + 2\pi \cdot /\delta)^{-j}$, and $\chi(x) = 2\pi i e^{2\pi icx}(e^{2\pi ix} - 1)^{-1}$. Now letting $q_j(\lambda, c) = [e_c m_j]_\delta(\lambda)$ and using the product rule, it is easy to see that the following recursion applies: $q_{j+1}(\lambda, c) = -\frac{1}{j}\left(\frac{\partial}{\partial\lambda}q_j(\lambda, c) - i\delta c q_j(\lambda, c)\right)$. Starting with $q_2(\lambda, c)$ computed through brute force, we can use this recursion to obtain these functions for $j = 2, 3, 4, 5, 6$, which are all needed in Section 4. Using the shorthand $z = e^{-i\lambda\delta}$, we have

$$[e_c m_2]_\delta(\lambda) = \frac{\delta(1 + c(\bar{z} - 1))}{|1 - z|^2},$$

$$[e_c m_3]_\delta(\lambda) = \frac{i^3\delta^2}{2}\left((\bar{z} - z)|1 - z|^{-4} - c^2(z - 1)^{-1} - 2c|1 - z|^{-2}\right),$$

$$\begin{aligned}[e_c m_4]_\delta(\lambda) = \frac{\delta^3}{6}\big((z + \bar{z} + 4)|1 - z|^{-4} - c^3(z - 1)^{-1} - 3c^2|1 - z|^{-2} \\ + 3c(\bar{z} - z)|1 - z|^{-4}\big),\end{aligned}$$

$$[e_c m_5]_\delta(\lambda) = \frac{i^5\delta^4}{24}\left[\begin{array}{l}\dfrac{c^4}{z - 1} + \dfrac{4c^3}{|1 - z|^2} + \dfrac{6c^2(z - \bar{z})}{|1 - z|^4} + \dfrac{8c(z - \bar{z})^2}{|1 - z|^6} + \dfrac{4c(z + \bar{z})}{|1 - z|^4} \\ + \dfrac{6(z - \bar{z})^3}{|1 - z|^8} + \dfrac{6(z^2 - \bar{z}^2)}{|1 - z|^6} + \dfrac{z - \bar{z}}{|1 - z|^4}\end{array}\right],$$

$$[e_c m_6]_\delta(\lambda) = \frac{\delta^5}{120}\left[\begin{array}{l}\dfrac{c^5}{z - 1} + \dfrac{5c^4}{|1 - z|^2} + \dfrac{10c^3(z - \bar{z})}{|1 - z|^4} + \dfrac{20c^2(z - \bar{z})^2}{|1 - z|^6} \\ + \dfrac{10c^2(z + \bar{z})}{|1 - z|^4} + \dfrac{30c(z - \bar{z})^3}{|1 - z|^8} + \dfrac{30c(z^2 - \bar{z}^2)}{|1 - z|^6} \\ + \dfrac{5c(z - \bar{z})}{|1 - z|^4} + \dfrac{24(z - \bar{z})^4}{|1 - z|^{10}} + \dfrac{36(z - \bar{z})^2(z + \bar{z})}{|1 - z|^8} \\ + \dfrac{z + \bar{z}}{|1 - z|^4} + \dfrac{12(z^2 + \bar{z}^2) + 2(z - \bar{z})^2}{|1 - z|^6}\end{array}\right].$$

A.2. Proofs

Proof of Theorem 1. The following construction will be used in the proof: define the continuous-time process $\bar{w}(t) = (1 - B)^d[\mathcal{I}^d w](t)$, with $\bar{w}(\delta\tau) = w_\tau$ in the stock case. This is easily shown to be stationary with autocovariance function given by

$$\frac{1}{2\pi}\int_{-\infty}^{\infty}\left|\frac{1 - e^{-i\lambda\delta}}{i\lambda}\right|^{2d} f_w(\lambda)e^{i\lambda h}\,d\lambda. \tag{A.1}$$

Incidentally, this demonstrates the stationarity of (7). Next, for any $t = \delta\tau + \delta c$, the filter discretization error process is $\epsilon_\tau = \Psi_c(B)y_\tau - x(t)$; it suffices to show that this error process is orthogonal to the available data

Y. We first demonstrate that this error process is stationary with mean zero. Consider that y_τ is a stock (3); then the polynomial term in (6) under application of the filter $\Psi_c(B)$ is simply

$$\sum_{j=0}^{d-1} \frac{\delta^j}{j!} y^{(j)}(0) \sum_k \psi_k(c)(\tau - k)^j = \sum_{j=0}^{d-1} \frac{y^{(j)}(0)}{i^j j!} \frac{\partial^j}{\partial \lambda^j} \left(\Psi_c(e^{-i\delta\lambda}) e^{i\lambda\delta\tau} \right) |_{\lambda=0}.$$

At the same time, the polynomial term in $x(t)$ is, using (5),

$$\sum_{j=0}^{d-1} \frac{y^{(j)}(0)}{j!} \int \psi(v)(t-v)^j \, dv = \sum_{j=0}^{d-1} \frac{y^{(j)}(0)}{j!} \int \psi(v + \delta c)(\delta\tau - v)^j \, dv$$
$$= \sum_{j=0}^{d-1} \frac{y^{(j)}(0)}{i^j j!} \frac{\partial^j}{\partial \lambda^j} \left(g(\lambda) e_{c+\tau}(\lambda) \right) |_{\lambda=0}.$$

So in order for the polynomial terms in the expression for ϵ_τ to cancel out, it is sufficient that $(ge_c)^{(k)}(0) = \Psi_c^{(k)}(0)$ for all $k < d$, where by $\Psi_c^{(k)}(0)$ we denote the kth derivative with respect to λ of the frf $\Psi_c(e^{-i\lambda\delta})$, evaluated at $\lambda = 0$. It turns out this condition is true for any $j < 2d$. Define the function

$$p_j(\lambda) = f_w m_j [f_w m_j]_\delta^{-1}(\lambda) = \delta \left(1 + \sum_{l \neq 0} \frac{f_w(\lambda + 2\pi l/\delta)}{f_w(\lambda)} \left(\frac{\lambda}{\lambda + 2\pi l/\delta} \right)^j \right)^{-1}.$$

Since f_w has no zeroes or poles, it can be shown that $p_j(0) = \delta$ and $p_j(2\pi l/\delta) = 0$ if $l \neq 0$. Also the first $j-1$ derivatives of p_j are all zero at $2\pi l/\delta$ for any integer l; this requires the existence of the first $j-1$ derivatives of f_w. Now using the stated formula in Theorem 1, $\Psi_c(e^{-i\lambda\delta}) = [ge_c p_{2d}]_\delta(\lambda)$. It follows that $\frac{\partial^k}{\partial\lambda^k} \Psi_c(e^{-i\lambda\delta})|_{\lambda=0} = (ge_c)^{(k)}(0)$ for all $k \leq 2d-1$, which is obtained by expanding the fold of $\Psi_c(e^{-i\lambda\delta})$, use of the product rule, and use of the above-stated properties of p_{2d}.

In the flow case, let $\Theta_c(B) = (1-B)\Psi_c(B)$; then the polynomial term in (10) under application of the filter $\Psi_c(B)$ is simply

$$\sum_{j=0}^{d-1} \frac{y^{(j)}(0)}{(j+1)!} \left(\sum_k \theta_k(c)(\delta\tau - \delta k)^{j+1} \right)$$
$$= \sum_{j=0}^{d-1} \frac{y^{(j)}(0)}{i^{j+1}(j+1)!} \frac{\partial^{j+1}}{\partial \lambda^{j+1}} \left(\Theta_c(e^{-i\delta\lambda}) e^{i\lambda\delta\tau} \right) \Big|_{\lambda=0},$$

where $\theta_k(c)$ are the coefficients of $\Theta_c(B)$. So in order for the polynomial terms in the expression for ϵ_τ to cancel out, it is sufficient that $(ge_{c+\tau})^{(j)}(0) = (\Theta_c(e^{-i\lambda\delta})e^{i\lambda\delta\tau})^{(j+1)}(0)/(i(j+1))$ for every τ. Using the stated formula for the flow frf in Theorem 1, we have $\Theta(e^{-i\lambda\delta})/i = [ge_c m_1^{-1} p_{2d+2}]_\delta(\lambda)$, whose kth derivative at $\lambda = 0$ is equal to $k(ge_c)^{(k-1)}(0)$ (or zero if $k = 0$). Then

$$\frac{1}{j+1}\sum_{k=0}^{j+1}\binom{j+1}{k}k(ge_c)^{(k-1)}(0)(i\delta\tau)^{j+1-k}$$
$$= \sum_{k=0}^{j}\binom{j}{k}(ge_c)^{(k)}(0)(i\delta\tau)^{j-k} = (ge_{c+\tau})^{(j)}(0),$$

as desired. Now returning to the stock case, our error process is

$$\epsilon_\tau = \left(\Psi_c(B) - \Psi(L)L^{-\delta c}\right)[\mathcal{I}^d w](\delta\tau). \tag{A.2}$$

In this formula, $\Psi_c(B)$ operates on $[\mathcal{I}^d w]$ as a discrete filter on a discrete (stock-observed) process, whereas $\Psi(L)L^{-\delta c}$ operates on $[\mathcal{I}^d w]$ as a continuous-lag filter on a continuous time process; we have shown above that $\Psi_c(B) - \Psi(L)L^{-\delta c}$ annihilates polynomials of degree $d-1$ in an integer variable τ. Now since $\bar{w}(t) = (1-B)^d[\mathcal{I}^d w](t)$, we can apply Lemma 1 of Bell (1984) to obtain a representation

$$[\mathcal{I}^d w](\delta\tau) = \sum_{j=1}^{d} A_{j,\tau+d}[\mathcal{I}^d w](\delta(j-d)) + \Xi_\tau(B)\bar{w}(\delta\tau), \tag{A.3}$$

where $A_{j,\tau}$ is a deterministic coefficient sequence dependent on time τ, which is completely determined by the differencing polynomial $(1-B)^d$. Also, the time-dependent discrete filter $\Xi_\tau(B)$ is given by the formula

$$\Xi_\tau(B)(1-B)^d = 1 - \sum_{j=1}^{d} A_{j,\tau+d}B^{\tau+d-j}. \tag{A.4}$$

The coefficient sequences $A_{j,\tau}$ consist of polynomials in τ of degree at most $d-1$. Moreover, since $\bar{w}(t)$ is a stationary process with integrable spectral density, it is stochastically continuous and thus has an orthogonal increments representation $\bar{w}(t) = \int_{-\infty}^{\infty} e^{i\lambda t}d\zeta(\lambda)$ – see Theorem 4.11.1 of Priestley (1981). Also, from (A.1) we know that $\mathbb{E}|d\zeta(\lambda)|^2 = |1 - e^{-i\lambda\delta}|^{2d}\lambda^{-2d}f_w(\lambda)d\lambda$; then by (A.4)

$$\Xi_\tau(B)\bar{w}(\delta\tau) = \int_{-\infty}^{\infty}\frac{e^{i\lambda\delta\tau} - \sum_{j=1}^{d} A_{j,\tau+d}e^{-i\lambda\delta(d-j)}}{(1-e^{-i\lambda\delta})^d}\,d\zeta(\lambda),$$

with the integrand being bounded. Putting this together with (A.2) and (A.3) yields $\epsilon_\tau = \int_{-\infty}^{\infty} \frac{e^{i\lambda\delta\tau}\left(\Psi_c(e^{-i\lambda\delta}) - g(\lambda)e_c(\lambda)\right)}{(1-e^{-i\lambda\delta})^d}\, d\zeta(\lambda)$. This shows that the error process is stationary with mean zero. Moreover, (A.2) shows that the error process is orthogonal to y^* under condition (9)—but note that (8) is not sufficient in general. It only remains to show that the error process is orthogonal to $\{w_\tau\}$:

$$\begin{aligned}\mathbb{E}[\epsilon_\tau w_{\tau+h}] &= \int_{-\infty}^{\infty} e^{-i\lambda\delta h}\left(\Psi_c(e^{-i\lambda\delta}) - g(\lambda)e_c(\lambda)\right)(1-e^{i\lambda\delta})^d \lambda^{-2d} f_w(\lambda)\, d\lambda \\ &= \delta \int_{-\pi/\delta}^{\pi/\delta} e^{-i\lambda\delta h}\left(\Psi_c(e^{-i\lambda\delta})[m_{2d}f_w]_\delta(\lambda) - [m_{2d}ge_cf_w]_\delta(\lambda)\right)(1-e^{i\lambda\delta})^d\, d\lambda,\end{aligned}$$

using the property of folds, and the periodicity of the frf of $\Psi_c(B)$. Now the stated stock formula in the theorem makes this covariance identically zero for all τ and h.

Finally, we turn to the flow case; the argument is essentially the same, with a few computational differences. Now letting $\bar{w}(t) = (1-B)^{d+1}[\mathcal{I}^{d+1}w](t)$, we note that by (11) $\bar{w}(\delta\tau) = w_\tau$; moreover, $\bar{w}(t)$ is stochastically continuous with integrable spectral density, so it has an orthogonal increments representation $\int e^{i\lambda t}d\zeta(\lambda)$, with $\mathbb{E}|d\zeta(\lambda)|^2 = |1-e^{-i\lambda\delta}|^{2d+2}\lambda^{-2d-2}f_w(\lambda)\,d\lambda$. The analog of (A.2) is now $\epsilon_\tau = \left(\Theta_c(B) - \Psi(L)L^{-\delta c}D\right)[\mathcal{I}^{d+1}w](\delta\tau)$, with similar interpretations of the discrete and continuous-lag filters (recall that D acts by differentiation, mapping $[\mathcal{I}^{d+1}w]$ to $[\mathcal{I}^{d}w]$). Moreover, the expression in parentheses annihilates polynomials of degree d in an integer variable τ, which is implicit in the earlier flow calculations of this proof. So we obtain that y^* is orthogonal to the error process under (9). We can also find a representation for the nonstationary process $[\mathcal{I}^{d+1}w](\delta\tau)$ exactly analogous to the stock case; the upshot is that $\epsilon_\tau = \int_{-\infty}^{\infty} \frac{e^{i\lambda\delta\tau}\left(\Theta_c(e^{-i\lambda\delta}) - g(\lambda)e_c(\lambda)i\lambda\right)}{(1-e^{-i\lambda\delta})^{d+1}}\, d\zeta(\lambda)$. (Recall that the frf of D is $i\lambda$.) Finally,

$$\begin{aligned}\mathbb{E}[\epsilon_\tau w_{\tau+h}] &= \int_{-\infty}^{\infty} e^{-i\lambda\delta h}\left(\Theta_c(e^{-i\lambda\delta}) - g(\lambda)e_c(\lambda)i\lambda\right)(1-e^{i\lambda\delta})^{d+1}\lambda^{-2d-2}f_w(\lambda)\, d\lambda \\ &= \delta\int_{-\pi/\delta}^{\pi/\delta} e^{-i\lambda\delta h}\Big(\Theta_c(e^{-i\lambda\delta})[m_{2d+2}f_w]_\delta(\lambda) \\ &\qquad - i[m_{2d+1}ge_cf_w]_\delta(\lambda)\Big)(1-e^{i\lambda\delta})^{d+1}\, d\lambda,\end{aligned}$$

which is identically zero by definition of $\Psi_c(B)$ and $\Theta_c(B)$. This completes the proof. □

Proof of Theorem 2. This proof follows along the same lines as Theorem 1, though it is not a corollary. We first consider the case of a stock; then by assumption n is stochastically continuous and thus can be written as $n(t) = \int e^{it\lambda} d\zeta_n(\lambda)$ with f_n integrable. The error process is then $\epsilon_\tau = \Psi_c(B)y_\tau - s(t) = \Psi_c(B)n_\tau + \left(\Psi_c(B) - L^{-\delta c}\right) s_\tau$. Now applying the machinery of the proof of Theorem 1 to the process s_τ, we find that the operator $\Psi_c(B) - L^{-\delta c}$ annihilates polynomials of degree $d-1$ so long as $\Psi_c^{(k)}(0) = (e_c)^{(k)}(0)$ for $k < d$; essentially we substitute $g \equiv 1$ in the calculations in the proof of Theorem 1. We likewise have $\Psi_c(e^{-i\lambda\delta}) = [e_c f_u p_{2d}/f_w]_\delta(\lambda)$, and it follows that $\frac{\partial^k}{\partial\lambda^k}\Psi_c(e^{-i\lambda\delta})|_{\lambda=0} = \sum_{l=0}^k \binom{k}{l} e_c^{(k-l)}(0)\left(\frac{f_u}{f_w}\right)^l(0) = (e_c)^{(k)}(0)$ for all $k \leq 2d-1$. The last equality follows from the fact that the first $2d-1$ derivatives of f_u/f_w are zero at $\lambda = 0$, but the value of the function at zero is unity. Applying this result, we see that ϵ_τ is stationary; using techniques from the proof of Theorem 1, we easily obtain the representation $\epsilon_\tau = \int e^{i\lambda\delta\tau}\Psi_c(e^{-i\lambda\delta})\, d\zeta_n(\lambda) + \int e^{i\lambda\delta\tau} \frac{\Psi_c(e^{-i\lambda\delta}) - e_c(\lambda)}{(1-e^{-i\lambda\delta})^d}\, d\zeta_u(\lambda)$, where $\mathbb{E}|d\zeta_u(\lambda)|^2 = |1 - e^{-i\lambda\delta}|^{2d}\lambda^{-2d} f_u(\lambda)\, d\lambda$. Likewise, we have

$$w_\tau = u_\tau + (1-B)^d n_\tau = \int e^{i\lambda\delta\tau}\, d\zeta_u(\lambda) + \int e^{i\lambda\delta\tau}(1 - e^{-i\lambda\delta})^d\, d\zeta_n(\lambda).$$

Now for optimality, it suffices to show that y^* and $\{w_\tau\}$ are orthogonal to the error process; in order for $\mathbb{E}[y^*\epsilon_\tau] = 0$, we need to assume (16). Then we have

$$\begin{aligned}
\mathbb{E}[\epsilon_\tau w_{\tau+h}] &= \int e^{-i\lambda\delta h}\Psi_c(e^{-i\lambda\delta})(1 - e^{i\lambda\delta})^d f_n(\lambda)\, d\lambda \\
&\quad + \int e^{-i\lambda\delta h}\frac{\Psi_c(e^{-i\lambda\delta}) - e_c(\lambda)}{(1 - e^{-i\lambda\delta})^d}\, |1 - e^{-i\lambda\delta}|^{2d}\lambda^{-2d} f_u(\lambda)\, d\lambda \\
&= \int e^{-i\lambda\delta h}(1 - e^{i\lambda\delta})^d \left(\Psi_c(e^{-i\lambda\delta}) f_w(\lambda) - e_c(\lambda) f_u(\lambda)\right)\lambda^{-2d}\, d\lambda \\
&= \delta\int_{-\pi/\delta}^{\pi/\delta} e^{-i\lambda\delta h}(1 - e^{i\lambda\delta})^d \left(\Psi_c(e^{-i\lambda\delta})[f_w m_{2d}]_\delta(\lambda) - [e_c f_u m_{2d}]_\delta(\lambda)\right) d\lambda,
\end{aligned}$$

which is identically zero by the stated formula for the frf of $\Psi_c(B)$. Turning to the case of a flow, we no longer have n stochastically continuous, but we can still write $n_\tau = \int e^{i\lambda\delta\tau}\, d\zeta_n(\lambda)$ with $\mathbb{E}|d\zeta_n(\lambda)|^2 = |1 - e^{-i\lambda\delta}|^2\lambda^{-2} f_n(\lambda)\, d\lambda$, since the flow-sampling of n, considered as a continuous-time process, will be in $\mathscr{C}^0$. As for $\Psi_c(B) - L^{-\delta c}$ operating on s_τ, this annihilates polynomials and reduces the signal to stationarity—simply use the concepts from the proof of Theorem 1. Then we obtain $\epsilon_\tau = \int e^{i\lambda\delta\tau}\Psi_c(e^{-i\lambda\delta})\, d\zeta_n(\lambda) + \int e^{i\lambda\delta\tau}\frac{\Theta_c(e^{-i\lambda\delta}) - e_c(\lambda) i\lambda}{(1-e^{-i\lambda\delta})^{d+1}}\, d\zeta_u(\lambda)$, with $\mathbb{E}|d\zeta_u(\lambda)|^2 = |1 - e^{-i\lambda\delta}|^{2d+2}\lambda^{-2d-2} f_u(\lambda)\, d\lambda$.

Hence y^* is orthogonal to the error process under (16), and

$$\begin{aligned}\mathbb{E}[\epsilon_\tau w_{\tau+h}] &= \int e^{-i\lambda\delta h}\Psi_c(e^{-i\lambda\delta})(1-e^{i\lambda\delta})^d|1-e^{-i\lambda\delta}|^2\lambda^{-2}f_n(\lambda)\,d\lambda \\ &\quad + \int e^{-i\lambda\delta h}\frac{\Theta_c(e^{-i\lambda\delta}) - e_c(\lambda)i\lambda}{(1-e^{-i\lambda\delta})^{d+1}}|1-e^{-i\lambda\delta}|^{2d+2}\lambda^{-2d-2}f_u(\lambda)\,d\lambda \\ &= \int e^{-i\lambda\delta h}(1-e^{i\lambda\delta})^{d+1}\left(\Theta_c(e^{-i\lambda\delta})f_w(\lambda)\lambda^{-2d-2} - ie_c(\lambda)f_u(\lambda)\lambda^{-2d-1}\right)d\lambda \\ &= \delta\int_{-\pi/\delta}^{\pi/\delta} e^{-i\lambda\delta h}(1-e^{i\lambda\delta})^{d+1}\left(\Theta_c(e^{-i\lambda\delta})[f_w m_{2d+2}]_\delta(\lambda) - i[e_c f_u m_{2d+1}]_\delta(\lambda)\right)d\lambda.\end{aligned}$$

This is identically zero by the stated formulas for the frf, and this concludes the proof. □

Proof of Proposition 1. We first derive the following property:

$$\begin{aligned}\delta\int_{-\infty}^{\infty}\psi(\delta z)e_{c-z}(\lambda+2\pi j/\delta)\,dz &= e_c(\lambda+2\pi j/\delta)\delta\int_{-\infty}^{\infty}\psi(\delta z)e_{-z}(\lambda+2\pi j/\delta)\,dz \\ &= e_c(\lambda+2\pi j/\delta)g(\lambda+2\pi j/\delta)\end{aligned}$$

for any integer j. Thus, using this relation we can easily show, for stock or for flow, that $\Psi_c(e^{-i\lambda\delta}) = \delta\int_{-\infty}^{\infty}\psi(\delta z)\Phi_{c-z}(e^{-i\lambda\delta})\,dz$. This is obtained by using the explicit formulas in Theorem 1 and Corollary 1. Now $\Phi_{c-z}(e^{-i\lambda\delta}) = \sum_k \phi_0(c-z+k)e^{-i\lambda\delta k}$, utilizing (17). Hence we have

$$\begin{aligned}\Psi_c(e^{-i\lambda\delta}) &= \sum_k \delta\int_{-\infty}^{\infty}\psi(\delta z)\phi_0(c-z+k)e^{-i\lambda\delta k}\,dz \\ &= \sum_k \delta\int_{-\infty}^{\infty}\psi(\delta z+\delta k)\phi_0(c-z)\,dz e^{-i\lambda\delta k}.\end{aligned}$$

Now Fourier Inversion concludes the proof. □

A.3. Derivations for Section 4

Letting $\Psi_c(z)$ denote the desired frf, we have $\psi_j = \frac{\delta}{2\pi}\int_{-\pi/\delta}^{\pi/\delta}\Psi_c(e^{-i\lambda\delta})e^{i\lambda\delta j}\,d\lambda$; thus by change of variable the following formula:

$$\psi_j = \frac{1}{2\pi i}\begin{cases}\int_\Omega \Psi_c(x^{-1})x^{j-1}\,dx & j \geq 0 \\ \int_\Omega \Psi_c(x)x^{-j-1}\,dx & j \leq 0.\end{cases} \tag{A.5}$$

Here Ω denotes the unit circle. Note that when $j = 0$, we can apply either case as we see fit. The formula $\Psi_c(x)$ is obtained by substituting the variable x everywhere for z (and x^{-1} for $\bar{z}$), whereas for $\Psi_c(x^{-1})$ we do the opposite. Now by considering the analytic extension of Ψ_c to the unit disk, the above integrals are computed by calculating the sum of the residues at the poles of the integrand occurring within the unit disk (see Henrici, 1974, pp. 249–250).

Derivations for the LLM. Using the formulas derived at the end of A.1, the formulas for the frequency response functions are immediate. To get the coefficients, we express $\Psi_c(x)$ as a rational function:

$$\Psi_c(x) = \frac{\delta^2\left((x^3+4x^2+x)+c^3(1-x)^3+3c^2x(1-x)^2+3cx(1-x^2)\right)}{x\left(\delta^2(x^2+4x+1)-6(1-x)^2/q\right)},$$

$$\Psi_c(x) = \frac{3\delta\left((x^2+x)+c^2(1-x)^2+2cx(1-x)\right)}{x\left(\delta^2(x^2+4x+1)-6(1-x)^2/q\right)}.$$

for the flow-signal and stock-signal cases, respectively. So there is a simple pole at the origin; for the other poles we must analyze the quadratic $\delta^2(x^2+4x+1)-6(1-x)^2/q$. It follows from the quadratic formula that the roots are η_1, η_2 given by

$$\eta_1 = \frac{-(2\delta^2+6/q)+\sqrt{3\delta^2(\delta^2+12/q)}}{\delta^2-6/q}$$

$$\eta_2 = \frac{-(2\delta^2+6/q)-\sqrt{3\delta^2(\delta^2+12/q)}}{\delta^2-6/q}.$$

The signs of these roots depend upon the sign of $\delta^2 - 6/q$, which we assume is negative since generally both δ and q are small (see McElroy and Trimbur, 2006); if $\delta^2 - 6/q \geq 0$, the signs of the roots are flipped, but all formulas remain the same. Then it is easy to see that $0 < \eta_1 < 1$ and $\eta_2 > 1$. Now making use of (A.5), we must also compute $\Psi_c(1/x)$, which is expressed as a rational function below:

$$\Psi_c(1/x) = \frac{\delta^2\left((x^2+4x+1)-c^3(1-x)^3+3c^2(1-x)^2-3c(1-x^2)\right)}{\left(\delta^2(x^2+4x+1)-6(1-x)^2/q\right)},$$

$$\Psi_c(1/x) = \frac{3\delta x\left((x+1)+c^2(1-x)^2-2c(1-x)\right)}{\delta^2(x^2+4x+1)-6(1-x)^2/q},$$

for the flow-signal and stock-signal cases, respectively. So there is no pole at the origin, but there are two simple poles at η_1, η_2 just as for $\Psi_c(x)$. So for the flow-signal case, we must compute the following residues to get the coefficients:

$$\begin{aligned}
\psi_j &= Res(\Psi_c(x)x^{-j-1}, \eta_1) \qquad j \leq -2 \\
\psi_{-1} &= Res(\Psi_c(x), \eta_1) + Res(\Psi_c(x), 0) \\
\psi_0 &= Res(\Psi_c(1/x)x^{-1}, \eta_1) + Res(\Psi_c(1/x)x^{-1}, 0) \\
\psi_j &= Res(\Psi_c(1/x)x^{j-1}, \eta_1) \qquad j \geq 1.
\end{aligned}$$

These residues are easy to calculate, and we obtain the stated formulas for the coefficients. The normalization constants are given by

$$\begin{aligned}
r_-^f &= \frac{\delta^2\left(c^3(1-\eta_1)^3 + 3c^2(1-\eta_1)^2\eta_1 + 3c(1-\eta_1^2)\eta_1 + (\eta_1^3 + 4\eta_1^2 + \eta_1)\right)}{(\delta^2 - 6/q)(\eta_1 - \eta_2)} \\
r_+^f &= \frac{\delta^2\left(c^3(\eta_1 - 1)^3 + 3c^2(1-\eta_1)^2 + 3c(\eta_1^2 - 1) + (\eta_1^2 + 4\eta_1 + 1)\right)}{(\delta^2 - 6/q)(\eta_1 - \eta_2)}.
\end{aligned}$$

For the stock-signal case, the same formulas apply (although the residues themselves are slightly different) except in the case $j = 0$, since there is no pole at the origin; instead, $\psi_0 = Res(\Psi_c(1/x)x^{-1}, \eta_1)$. Now the normalization constants are given by

$$\begin{aligned}
r_-^s &= \frac{3\delta\left(c^2(1-\eta_1)^2 + 2c(1-\eta_1)\eta_1 + (\eta_1^2 + \eta_1)\right)}{(\delta^2 - 6/q)(\eta_1 - \eta_2)} \\
r_+^s &= \frac{3\delta\left(c^2(1-\eta_1)^2 - 2c(1-\eta_1) + (1+\eta_1)\right)}{(\delta^2 - 6/q)(\eta_1 - \eta_2)}.
\end{aligned}$$

Derivations for the STM and TP. The frequency response formulas follow immediately from the formulas derived in A.1. To get the coefficients, we express $\Psi_c(x)$ as a rational function:

$$\Psi_c(x) = \frac{\Theta_c(x)}{x\left(\delta^4(x^4 + 26x^3 + 66x^2 + 26x + 1) - 120(1-x)^4/q\right)},$$

noting that $\Theta_c(x)$ has a different definition for the flow-signal and stock-signal cases. This rationalization is obtained by multiplying numerator and denominator by x^3. Likewise for $\Psi_c(1/x)$, we have

$$\Psi_c(1/x) = \frac{\Phi_c(x)}{\delta^4(x^4 + 26x^3 + 66x^2 + 26x + 1) - 120(1-x)^4/q}$$

for either of the cases, obtained by multiplying numerator and denominator by x^2. For the flow-signal case (18) these functions are given explicitly by

$$\begin{aligned}\Theta_c(x) &= \delta^4(c^5(1-x)^5 + 5c^4(1-x)^4x + 10c^3(1-x^2)(1-x)^2 + 20c^2x(1-x^2)^2 \\ &\quad - 10c^2(1-x)^2(1+x^2) + 30cx(1-x^2)(1+x)^2 - 30cx(1-x^4) \\ &\quad + 5cx(1-x)^2(1-x^2) + x\left(x^4 + 26x^3 + 66x^2 + 26x + 1\right)) \\ \Phi_c(x) &= \delta^4(c^5(1-x)^5 + 5c^4(1-x)^4 - 10c^3(1-x^2)(1-x)^2 + 20c^2(1-x^2)^2 \\ &\quad - 10c^2(1-x)^2(1+x^2) - 30c(1-x^2)(1+x)^2 + 30c(1-x^4) \\ &\quad - 5c(1-x)^2(1-x^2) + \left(x^4 + 26x^3 + 66x^2 + 26x + 1\right)),\end{aligned}$$

whereas for the stock-signal case they are given by

$$\begin{aligned}\Theta_c(x) &= 5\delta^3(c^4(1-x)^4 + 4c^3(1-x)^3x + 6c^2x(1+x)(1-x)^2 \\ &\quad + 8cx(1+x)^2(1-x) - 4cx(1-x)(1+x^2) + 12x^2(1+x)) \\ \Phi_c(x) &= 5\delta^3(c^4x(1-x)^4 - 4c^3x(1-x)^3 + 6c^2x(1+x)(1-x)^2 \\ &\quad - 8cx(1-x)(1+x)^2 + 4cx(1-x)(1+x^2) + 12x^2(1+x)).\end{aligned}$$

So $\Psi_c(x)$ has a simple pole at the origin, whereas $\Psi_c(1/x)$ does not. For their other poles, we must consider the quartic function $\delta^4(x^4 + 26x^3 + 66x^2 + 26x + 1) - 120(1-x)^4/q$. We employ the following factorization:

$$\begin{aligned}&\delta^4(x^4 + 26x^3 + 66x^2 + 26x + 1) - 120(1-x)^4/q \\ &\quad = (\delta^4 - 120/q)\left(x^4 + bx^3 + dx^2 + bx + 1\right) \\ &\quad = (\delta^4 - 120/q)(x^2 + \alpha x + 1)(x^2 + \beta x + 1),\end{aligned}$$

where $b = (26\delta^4 + 480/q)/(\delta^4 - 120/q)$ and $d = (66\delta^4 - 720/q)/(\delta^4 - 120/q)$. Note that the symmetry in the quartic allows us to proceed without recourse to Cardano's formula. Now in the second equality, α and β are given by

$$\begin{aligned}\alpha &= \frac{b - \sqrt{b^2 - 4d + 8}}{2} = \frac{26\delta^4 + 480/q + \sqrt{60(7\delta^8 + 960\delta^4/q)}}{2(\delta^4 - 120/q)} \\ \beta &= \frac{b + \sqrt{b^2 - 4d + 8}}{2} = \frac{26\delta^4 + 480/q - \sqrt{60(7\delta^8 + 960\delta^4/q)}}{2(\delta^4 - 120/q)}.\end{aligned}$$

Each of the two quadratics is then further factored using the quadratic formula, which yields

$$v_2 = \frac{-\alpha - \sqrt{\alpha^2 - 4}}{2}, \qquad v_3 = \frac{-\beta + \sqrt{\beta^2 - 4}}{2}, \qquad v_4 = \frac{-\beta - \sqrt{\beta^2 - 4}}{2},$$

and $v_1 = (-\alpha + \sqrt{\alpha^2 - 4})/2$. As in the LLM, we make the assumption that δ and q are small, such that $\delta^4 < 120/q$. Now from the formula for α, we can show that $\alpha < -2$, which implies that $v_1 > 1$. Then it follows that $0 < v_2 < 1$. As for the other roots, observe that $|\beta| < 2$ (after some more inequality calculations), so that

$$v_3 = \frac{-\beta + i\sqrt{4 - \beta^2}}{2} \qquad v_4 = \frac{-\beta - i\sqrt{4 - \beta^2}}{2}.$$

Hence they are complex conjugate, and their product is unity. So the coefficients are just given by sums of residues, where the poles under consideration are v_2, v_3, v_4, and zero. These are all simple poles, so the calculations are easy. Note that since the poles v_3, v_4 occur on the unit circle, we should multiply their contribution to the residue sum by 1/2 (Henrici, 1974). From here the determination of the coefficients is straightforward.

For the turning point calculations, use the notation$\tilde{}$ to denote the TP functions. Then we have $\widetilde{\Psi}_c(x) = (1 - x)\Psi_c(x)$, and hence $\widetilde{\Theta}_c(x) = (1 - x)\Theta_c(x)$. Thus for $j < 0$, we can utilize the same formulas as the stock-signal STM case, but with $\Theta_c(x)$ multiplied by $1 - x$. However, $\widetilde{\Psi}_c(1/x) = (x - 1)\Psi(1/x)/x$, which adds an additional pole at zero. This means that for $j \geq 1$, we can use the stock-signal STM coefficient ψ_{j-1}, but with $\Phi_c(x)$ multiplied by $x - 1$. For ψ_0 we obtain a pole of order 2 at the origin, so that the corresponding residue is

$$Res\left(\frac{\Theta_c(x)(1 - x)}{(\delta^4 - 120/q)x^2(x^4 + bx^3 + dx^2 + bx + 1)}, 0\right) = \frac{10\delta^3 c(1 - c^2)(3c + 2)}{\delta^4 - 120/q}.$$

This together with the other residues yields the stated formula.

Derivations for the BP and LP. For the BP filter, we first derive the autocovariance function for $c(t)$. Write

$$f_c(\omega) = \frac{r\sigma^2}{[\rho + i(\omega - \lambda_c)][\rho - i(\omega - \lambda_c)][\rho + i(\omega + \lambda_c)][\rho - i(\omega + \lambda_c)]},$$

where $\omega \in \mathbb{C}$ by analytic extension. The poles in $\mathbb{C}^+$ are, therefore, $\pm\lambda_c + i\rho$. So for $x \geq 0$, we have

$$\begin{aligned} R_c(x) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} f_c(\lambda) e^{i\lambda x}\, d\lambda = i \sum_{\zeta \in \mathbb{C}^+} Res(f_c e^{ix\cdot}, \zeta) \\ &= \frac{r\sigma^2}{8\rho\lambda_c} e^{-\rho x}\left(\frac{e^{ix\lambda_c}}{\lambda_c + i\rho} + \frac{e^{-ix\lambda_c}}{\lambda_c - i\rho}\right) \\ &= \frac{r\sigma^2 e^{-\rho x}}{4\rho\lambda_c(\lambda_c^2 + \rho^2)} Re\left(\frac{e^{ix\lambda_c}}{\lambda_c + i\rho}\right). \end{aligned}$$

Noting that R_c is an even function, we obtain the stated result. In order to obtain the spectrum of the flow-sampled cycle, we must compute $|1-z|^2[f_c m_2]_\delta$. The fold is

$$\begin{aligned} [f_c m_2]_\delta(\lambda) &= \frac{\delta r\sigma^2}{(\rho^2 + \lambda_c^2)^2 |1-z|^2} \\ &\quad - \frac{r\sigma^2}{8\lambda_c\rho}\left((\lambda_c + i\rho)^{-3}\frac{1 - e^{2\delta(\rho - i\lambda_c)}}{|z - e^{\delta(\rho - i\lambda_c)}|^2} + (\lambda_c - i\rho)^{-3}\frac{1 - e^{2\delta(\rho + i\lambda_c)}}{|z - e^{\delta(\rho + i\lambda_c)}|^2}\right), \end{aligned}$$

which is determined by using Theorem 4.9a of Henrici (1974). Alternatively, one can differentiate with respect to c the expressions given below, evaluating at $c = 0$. Now multiplying the above formula by $|1-z|^2$, we see that the spectrum corresponds to an ARMA(2,2) process. We next compute $[e^{ic\delta\cdot} f_c m_4]_\delta(\lambda)$. By Theorem 4.9b of Henrici (1974) we have the fold equal to $-\frac{1}{\delta}\sum_\zeta Res\left(f_c(\lambda + 2\pi\cdot/\delta)(\lambda + 2\pi\cdot/\delta)^{-4}\chi, \zeta\right) e^{i\delta c\lambda}$, where the sum is over all poles ζ. There is a pole of order 4 at $-\frac{\delta\lambda}{2\pi}$, and the residue is

$$\begin{aligned} &\frac{1}{6}\left(\frac{\delta}{2\pi}\right)^4\left(f_c(0)\dddot{\chi}\left(-\frac{\delta\lambda}{2\pi}\right) + 3\left(\frac{2\pi}{\delta}\right)^2 \ddot{f}_c(0)\dot{\chi}\left(-\frac{\delta\lambda}{2\pi}\right)\right) \\ &= \frac{1}{6}\left(\frac{\delta}{2\pi}\right)^4\left[\begin{array}{c} \frac{r\sigma^2}{(\rho^2+\lambda_c^2)^2}(2\pi i)^4 e^{-i\delta c\lambda}\left(\frac{c^3}{z-1} + \frac{3c^2}{|1-z|^2} + \frac{3c(z-\bar{z})}{|1-z|^4} + \frac{2(z-\bar{z})^2}{|1-z|^6} + \frac{z+\bar{z}}{|1-z|^4}\right) \\ + \left(\frac{2\pi}{\delta}\right)^2 \frac{12r\sigma^2(\lambda_c^2-\rho^2)}{(\rho^2+\lambda_c^2)^4}(2\pi i)^2 e^{-i\delta c\lambda}\left(\frac{c}{z-1} + \frac{1}{|1-z|^2}\right) \end{array}\right]. \end{aligned}$$

(The derivatives of χ are computed in McElroy and Trimbur, 2008.) There are also simple poles at $\frac{\delta}{2\pi}(-\lambda \pm \lambda_c \pm i\rho)$, whose residues are given below:

$$\begin{aligned} &-\frac{\delta}{2\pi i}\frac{(\lambda_c - i\rho)^{-5} r\sigma^2}{8\rho\lambda_c}\chi\left(\frac{\delta}{2\pi}(-\lambda + \lambda_c - i\rho)\right) \\ &\frac{\delta}{2\pi i}\frac{(\lambda_c + i\rho)^{-5} r\sigma^2}{8\rho\lambda_c}\chi\left(\frac{\delta}{2\pi}(-\lambda + \lambda_c + i\rho)\right) \end{aligned}$$

$$-\frac{\delta}{2\pi i}\frac{(\lambda_c + i\rho)^{-5} r\sigma^2}{8\rho\lambda_c}\chi\left(\frac{\delta}{2\pi}(-\lambda - \lambda_c - i\rho)\right)$$
$$\frac{\delta}{2\pi i}\frac{(\lambda_c - i\rho)^{-5} r\sigma^2}{8\rho\lambda_c}\chi\left(\frac{\delta}{2\pi}(-\lambda - \lambda_c + i\rho)\right).$$

Summing these and simplifying gives the stated result. Now we also note that

$$\frac{\partial}{\partial c}[e^{ic\delta\cdot} f_c m_4]_\delta(\lambda) = i\delta[e^{ic\delta\cdot} f_c m_3]_\delta(\lambda) \qquad \frac{\partial^2}{\partial c^2}[e^{ic\delta\cdot} f_c m_4]_\delta(\lambda) = -\delta^2[e^{ic\delta\cdot} f_c m_2]_\delta(\lambda),$$

which allows us to obtain the other folds fairly easily.

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